On the discovery of the saros

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Abstract. It is believed that in ancient times the most simple and affordable method of predicting eclipses was based on the saros, an approximation to the time period of the eclipse cycle. In many studies techniques for finding the saros through observations of eclipses were considered, but the verification shows failure of these methods. In this paper we propose an experimental method for detecting saros observing lunar eclipses. Saros could be detected by analyzing the repetition periods of partial lunar eclipses with the same phase. This requires regular observations of eclipses over $200 \div 300$ years at an average density of at least 30 observations per century. The observer should record the time of the eclipse, its phase and the position of the shadow on the disk of the Moon. The paper proposes an experimental method of determining the duration of draconic and anomalistic months. The knowledge of these values allows to obtain "the saros relation", which Ptolemy assigns to Babylonian astronomy. The determination of the duration of anomalistic month requires the astrolabe or another instrument, which allows to measure the motion of the Moon and time. It is shown that the exeligmos, the better approximation of the eclipce cycle, can be found experimentally through the observations of the solar eclipses too.
Key words: saros detection, finding the length of the lunar months, history of astronomy

Върху откриването на сароса

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Смята се, че в древността най-простият и сигурен метод за предсказване на затъмненията се е основавал на сароса, едно приближение на продължителността на цикъла на затъмненията. Методи за намиране на сароса чрез наблюдения на затъмнания се разглеждат в много изследвания, но те не издържат пректическата проверка. В тази статия се предлага един експериментален метод за определяне на сароса чрез наблюдаване на затъмнения. Саросът може да бъде намерен чрез анализ на повторяемостта на частничните лунни затъмнения с една и съща фаза. Това изисква регулярно наблюдение на затъмнения в течение на $200 \div 300$ години със средна плътност поне 30 наблюдения на век. Наблюдателят следва да записва времето на затъмнанието, неговата фаза и позицията на сянката върху лунния диск. Тази статия предлага експериментален метод за определяне на драконичния и аномлиситчния месеци. Познаването на техните стойности позволява получаването на "сарос отношението", което Птоломей приписва на вавилонската астрономия. Определянето на аномалистичния месец изисква астролаб или друг инструмент за измерване на движението на Луната и времето. Показано е, че екселигмосът, по-добрата апроксимация на цикъла на затъмненията, може също да бъде намерен експериментално, чрез наблюдения на слънчеви затъмнения.

1. Introduction

Any popular reference book or encyclopedia of astronomy provides information that saros is a period equal to the duration of $S = 6585$ days (18 years and 10.33 or 11.33 days), after which solar and lunar eclipses are repeated in the previous sequence. If in year T, in a certain geographical point an eclipse was recorded, the date of the next eclipse, which can be predicted by using the saros will be $T + S$. Because the saros contains fractional part of the day "1/3", not every predicted eclipse can be observed from the given geographical point. Triple saros (exeligmos) corresponds to almost a whole day and its use provides a better ability to predict eclipses, which can be observed.

Ancient astronomers used the term saros as numeral, which means the calendar period of 3600 years. It was used to express the so-called "great year" - the period after which the motion of the Sun, the Moon and the all planets is repeated. In this sense, saros was used by Burros (2 BC), Abiden (2 century AD), Sincellus (about 800 AD) and others [NEUG]. There are other periods neros and sossos which correspond to the durations of 600 and 60 years. They also were used to express a "great year".

According to Ptolemy [PTOL], in the modern sense saros was used in Babylon, as the repetition period of lunar eclipses. In addition, Ptolemy leads to the so-called "saros relation", which binds the duration of saros and four lunar months: 1 saros = 223 synodic months (return of phase) = 241 sidereal month (return of longitude) = 239 anomalistic months (return of perigee) = 242 draconic month (return of latitude). However, he did not specify when and how the saros was found, and how was found the "saros relation", which implies knowledge of the ancient astronomers of the duration of the lunar months.

The aim of this study is to find answers to next questions. In what way the period of the repetition eclipses $S = 6585.33$ days could be found, which we now call the saros? By what means could be determined the "saros relation"? What knowledge, astronomical instruments and observations are necessary in order to detect saros and the "saros relation"?

2. Attempts to search for the saros

One of the first attempts to explain the method of finding the saros belongs to A. Pannekoek [PANN]. His idea is as follows. Babylonian astronomers knew from observations that lunar eclipses occur in series. Usually a partial eclipse with a small phase begins a new series. In the next eclipses the phase of the eclipse is increasing, and the observer can register eclipses with a total phase and finally, it begins to decrease. Then a break follows, when the eclipses are not observed, after which begins a new series of eclipses. The author found that in the period from 750 to 650 BC there were a four series of 5 eclipses in row and a four series of 4 eclipses in a row, which were visible in Babylon. It is possible, such fortunate set of realizations of the observed eclipses promoted that an observers found that eclipses occur at series. This could lead to the emergence of the simplest methods of eclipses prediction.

Pannekoek wrote out six series of eclipses that occurred in succession, and he considered the distance of the Moon to the node for each eclipse. The result was that in the first and sixth series of eclipses, the distances begin to repeat itself, which means a repeating of the eclipses circumstances. In the study Pannekoek used precision calculations. The Babylonian astronomers had to be able to quite accurately measure the distance of the Moon to the node to make similar work. However, we do not know anything about the existence of such observations and the necessary astronomical instruments. In the Babylonian observations, which describe the position of the Moon and planets relative to the fixed stars, the distances are expressed in the "ammat" and "ubani" [PANN]. Analysis of these observations shows that they have relatively low accuracy and suggests that the observations are visual. Instead of the distances, the magnitudes of the phases of the eclipses can be used. Magnitudes of phases can be evaluated by the naked eye, although such assessment will be burdened by errors.

Nevertheless, the main problem in the Pannekoek's approach is that for any selected series of eclipses, about half of the eclipses can not be observed. All penumbral eclipses cannot be registered by a naked eye, thus they have to be excluded from consideration. In some cases, the Moon is below the horizon. As an example, we selected set of the eight series, with a maximum density of visible eclipses in the Babylon. For brevity, we excluded penumbral eclipses, which finish each series. For partial and total eclipses the kind of the eclipse phase is clarified. Eclipses, which may be observed in the Babylon (the Moon was above the horizon) are marked in bold. As a result, in chosen set there are three series, which contain five eclipses in a row, table 1.

N	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	17.02.683	14.08.683	07.02.682 P	**03.08.682** **P**	**28.01.681** **T**	**22.07.681** **T**	**16.01.680** **P**	**11.07.680** **P**
2	07.12.680	**02.06.679** **P**	26.11.679 P	**22.05.678** **T**	15.11.678 T	11.05.677 P	**03.11.677** **P**	30.04.676
3	01.04.676	24.09.676	21.03.675 P	14.09.675 T	10.03.674 **T**	**04.09.674** **P**	**27.02.673** P	23.08.673
4	18.01.672	13.07.672	**07.01.671** **P**	**02.07.671** **P**	**28.12.671** **T**	22.06.670 T	**17.12.670** **T**	**11.06.669** **P**
5	12.05.669	05.11.669	02.05.668 P	**25.10.668** **P**	**21.04.667** **T**	**15.10.667** **T**	**10.04.666** **T**	**04.10.666** **P**
6	29.02.665	24.08.665	**17.02.664** **P**	**14.08.664** **P**	07.02.663 T	**03.08.663** **T**	**28.01.662** **P**	23.07.662 P
7	18.12.662	12.06.661 P	**06.12.661** **P**	**02.06.660** **T**	**25.11.660** **T**	**22.05.659** **T**	**05.11.659** **P**	11.05.658
8	12.04.658	06.10.658	31.03.657 P	**24.09.657** **P**	**20.03.656** **T**	**14.09.656** **T**	10.03.655 P	03.09.655 P

Table 1. Eight series of eclipses occurred in the period from 683 to 655 BC.

Eclipses of series (1) do not coincide with the eclipses of series (6), the series (2) does not coincide with the series (7), and the series (3) does not coincide with (8). The situation deteriorates even more, if we take into account the possible omissions of eclipses because of weather conditions or of a small magnitude of phases. In addition, when drawing up the table, we used the knowledge of the order of the eclipses in a series and in saros. Ancient observers could compile a table of only of the registered eclipses, and as a result the different lines of the table may be shifted relative to each other. It also complicate the situation. So, to find saros by method of Pannekoek seems to be not possible.

Developing the Pannekoek's approach, V. Bronshten [BRON] excluded from consideration penumbral eclipses. He folded the duration of the five observed series of eclipses, added the four intervals between this series and added the time interval between the last eclipse in the fifth series and the first eclipse in the sixth series. As a result, the author received a period equal to 223 synodic months, which represents a saros. However, after Pannekoek, Bronshten, for some reason, did not take into account the circumstances of visibility of eclipses. If from an arbitrary cycle consisting of a series of eclipses we exclude unobserved eclipses, then after a saros we don't get a series of similar eclipses. This has already been demonstrated by table 1. And if the first and sixth series of the observed eclipses differ from each other, the periodicity disappears and with it the reason to fold the length of just five episodes. In addition, because of the omissions of some eclipses, the first partial eclipse in series (1) and (6) may not be observed. It also leads to extra errors. Therefore, the saros is not possible to find by Bronshten's method.

Regardless of Ptolemy, a knowledge of the saros in Babylonia is confirmed by deciphering the cuneiform texts LBAT 1414, LBAT 1415+1416+1417, LBAT 1419 and LBAT 1428 (the so-called "Saros Canon"). Saros Canon is a compilation of lunar eclipses from 747 to 315 BC [STEE1]. Eclipses are gathered in columns, each of them has 38 rows. Each row corresponds to the date on which an eclipse is possible (or EP = eclipse possibility). The cycle of 38 eclipses is equal to the length of 223 synodic months and represents a saros. After passing of saros (223 synodic months) the sequence of eclipses is repeated, therefore the following 38 eclipses are placed in another column.

Each column is divided by horizontal lines, which allows for allocating five series of eclipses and determining their structure. Since as the beginning of saros we can take each of the five series, the sequence may look like $8-7-8-7-8$, $7-8-7-8-8$, $8-7-8-8-7$, $7-8-8-7-8$ or $8-8-7-8-7$ [STEE2]. Knowledge of the saros scheme allows to make the prediction of the date on which the eclipse is possible, and knowledge of the saros cycle duration allows for sufficient accuracy to determine the time of the next eclipse in saros [BRAC]. However, these considerations do not permit either to identify the method of detecting saros, nor specify when the discovery could be made. It is obvious that all of the above cuneiform texts represent a compilation that could be done in 4^{th} century BC, and even in later times, when saros was already known. Knowledge of the scheme saros and the dates of actually observed eclipses allows to sort the eclipses by saroses and determine within each saros all the dates of the eclipse possibilities.

On the other hand, there are points of view according to which, saros was known for a long time. For example, Brown [BROW, p. 205] assumes that period saros was used to predict eclipses in 7^{th} century BC. His opinion is based on the text of 8502, which predicted an eclipse in 679 BC. In the description of this eclipse, the visibility of Jupiter was mentioned.

"All the things which have come to concern the land of Akkad... An eclipse of the Moon and Sun in month III will take place. These sings are of bad fortune for Akkad... and now, in this month IX, an eclipse will take place... and Jupiter will stand in its eclipse."

To predict the visibility of Jupiter during the eclipse ancient astronomers should be able to accurately determine the time at which an eclipse will occur. It is possible that in the 7^{th} century BC a period of 6 lunar months was used to predict an eclipses. However, this period does not allow to determine a time of the eclipse with the required accuracy, because it is subject to significant change due to the uneven motion of the Moon. Hence the author concludes that this eclipse was predicted by the saros cycle, probably via an eclipse on 15 November 697 BC. It is estimated that an eclipse of 26 November 679 BC could be observed in Babylon at the dawn of the Moon. Calculations show that Saturn was placed in half a degree from the Moon, but rising of the Jupiter occurred four hours later after rising of the Moon, when the eclipse was over. Consequently the eclipse described in the text of 8502 does not match the circumstances of the eclipse on 26 November 679 BC. Either we are wrong in the dating of this text, or in the interpretation of the text. Finally, even if the dating and the interpretation of the text 8502 are right, to predict this eclipse it is possible to use another long time period. For example, the eclipse on 26 November 679 BC was possible to predict by period of repetition of eclipses of 7973.3 days and eclipse on 26 January 700 BC. In any case, Brown's arguments are untenable.

Another argument of Brown shifts the date of saros detecting to the middle of the 8^{th} century BC. In the beginning of plates LBAT 1413 and LBAT 1414 there are so-called "strange numbers" (by Steele terminology) "$1,40$" and "1.50". If we assume that these figures represent the time, expressed in "us", they can characterize the deviation of the period of 223 lunar months of 6585 days. According to Brown, the existence of "strange numbers" is an evidence of using saros to predict eclipses. Brack-Bernsen and Steele don't agree from this point of view [BRAC, p. 187]. First, Brown's assumption means that there should exist regular observations of eclipses before 747 BC, but we do not have knowledge for such observations.
Secondly, it is not always clear to what eclipses "a strange numbers" are related. The authors suggest that the number "1.50" refers to the entire row of the eclipses, but not only to the first eclipse. Finally, the authors showed that "strange numbers" are more consistent with the estimated duration of saros in the later times. Therefore, Brown's point of view on the use of saros in the middle of the 8^{th} century BC is extremely unconvincing.

Thus, we have considered various ideas about the method of the detecting of saros and the time when this discovery could be made. Arguably, the Babylonian astronomers knew the saros and, from a certain time, it was used to predict eclipses. Moreover, the observations of previous centuries were systematized and were included in the saros scheme. However, so far no any workable method by which one could detect saros has been proposed.

3. Searching for the saros by frequency method

Statement of the problem. Let's consider the possibility of finding saros by the method based on the allocation of the most frequently recurring intervals (or quasiperiods) between eclipses. To do this, we choused a geographic point, and formed the canons of lunar and solar eclipses, which can be observed from this point during a sufficiently long time. We assume, weather conditions always allow observing the eclipse, and the observer can register partial eclipses with an arbitrarily small phase.
The searching algorithm of repetition periods of eclipses is formed as follows. Let us have a canon of eclipses. Let's take Julian date of the first eclipse in the list, subtract it from the dates of the remaining *n-1* eclipses and in result, we obtain a set of periods $t_{12} \div t_{1n}$. Next, we'll take the date of the second eclipse in the canon and subtract it in similar way from the rest of the dates of eclipses, and thereby we'll obtain a set of periods $t_{23} \div t_{2n}$, fig. 1.

Let's compare the length of the periods of the second set t_{2j}, (where $j \in [3; n]$) with the set of the first periods t_{1i}, (where $i \in [2; n]$). If for some numbers i and j the equality $t_{1i} = t_{2j}$ is correct, then the frequency of this period t_{1i} is increased per unit $f_i = f_i + 1$. Otherwise, the set of periods $t_{12} \div t_{1n}$ is complemented by period t_{2j}, $t_{1(n+1)} = t_{2j}$, which is credited with the frequency equal to one $f_{n+1} = 1$. Then, we will take the third eclipse in canon and repeat this procedure until all eclipses from the available list will be used. As a result, we obtain a set of periods $\{t\}$ and a corresponding set of frequencies $\{f\}$. Due to the uneven motion of the Moon in its orbit, short periods of recurrence of eclipse is prone to small changes. Therefore, when calculating the frequencies we demanded approximate implementation of the equality $t_{1i} \approx t_{2j}$.

Initial conditions. To construct a canon of eclipses, we used interval of $T = 300$ years, the duration of which corresponds to 16 saros. A point of observation is preferable to select in a range of latitudes $30 \div 40$ degrees. The choice of the observation point between the northern tropic and mid-latitudes is reasonable because here Greek, Chinese, Egyptian and Babylonian astronomers carried out their observations. As the observation point Cairo was chosen. In the result, we got the canon of 287 lunar eclipses and the canon of the 100 solar eclipses, which will be used in the model calculations.

Searching for the period by the model canons of eclipses. Applying the above algorithm to the canon of lunar and solar eclipses, we obtain the following results, fig. 2÷4. From fig. 2 follows that the frequency of eclipses is grouped in the form of packets with an average repetition period of about $\sim 1200 \div 1300$ days. Each package corresponds to some series of eclipses and it consists of several frequencies, separated by an interval of ~ 177 days. However, in this experiment it is impossible to separately highlight the frequency, which corresponds to the saros for all variants of calculation. Saros is not the dominant period even in its "frequency package" and it is lost among the multitude of frequencies of other periods. Among the longtime periods of recurrence of eclipses, exeligmos has the highest frequency, fig. 3. However exeligmos can not be confidently selected among other frequencies by the analyses of lunar eclipses. Fig. 4 shows that exeligmos could be found by observing solar eclipses. Even with the exclusion of the "weak" eclipses with magnitude of phase less than 0.60, its frequency is higher than frequencies of other periods by $2-3$ times. Saros was not found even in the first two dozens of the most intense periods, which have duration of less 7000^d. Consequently, it is impossible to find saros analyzing the observations of the solar eclipses.

Searching of the saros through the Chinese observations. To verify this result we used Chinese observations of the lunar and solar eclipses in the period from 3^{th} to 17^{th} centuries [XU]. In contrast to the model canons, the Chinese canons are more long-continued over time, but they have a much lower density of observations. The results of calculations suggest that the situation has not undergone a qualitative change, fig. 5, fig. 6.

Conclusion. The application of frequency method to the model canons and the real canons of eclipses revealed that it is impossible to find the saros among other periods of the recurrence of eclipses. According to observations of solar eclipses, exeligmos can be found experimentally, as it proved through an analyces of the Chinese observations.

4. Searching for the saros by modified frequency method

Modification of the algorithm. Let's apply the frequency analysis to the model canons of eclipses with one change. We will determine the frequency of periods using only a pair of eclipses, which had approximately the same phase. In this case we will use only partial eclipses, but all total eclipses must be excluded. In view of the fact that the Earth's shadow at about 2.5 times larger then the diameter of the moon, the total phase of the eclipse can usually be achieved at different distances to the node. Among the partial eclipses will use such eclipses, in which the observer had the opportunity to register the time of the maximum phase. That is, part of the eclipse, which occurred at rise or set of the Moon should be excluded from consideration.

Modern calculations allow to determine the magnitude of eclipse phase with very high accuracy. Ancient astronomers estimated phase visually, so all their estimates have some error. As a result, the eclipse at different phases can be evaluated as equal, and vice versa. Therefore, we need to define the characteristic value of the error of phase and take it into account in the frequency analysis. It is known, that the ancient astronomers estimated phases of the eclipses in "fingers", with total phase corresponded to 12 "fingers". Hence, we can find that one "finger" corresponds to the phase 0.08, which probably is the limit assessment for the naked eye.

The error of the phase of partial eclipses, calculated for 11 lunar eclipses Almagest was 0.56 ± 0.17 "fingers" with a maximum magnitude error of 1.68 "fingers". Note that the phase error distribution does not correspond to the Gaussian distribution. Given all of the above, we assume that the estimate of the error phase 1 point fairly accurately describes the error phase. In the case of comparing the phases of the two eclipses, the total error may be less than this value, if in the evaluation phase of each eclipse we are wrong in one direction, or more then this value, if vice versa. On average, the error will increase by $\sqrt{2}$ times and will be 1.41 "fingers", or $\varDelta\varphi = 0.11$. Thus, when calculating the frequencies of periods we will select a pair of eclipses of the model of the canon, if the condition $\varDelta\varphi = |\varphi_2 - \varphi_1| \leq 0.11$ is valid.

Searching of the saros by the model canon of eclipses. Result of application of the modified algorithm to the canon of lunar eclipses is shown below, fig. 7. The highest frequencies correspond to the exeligmos, the saros and the double saros. Given their multiplicity, it is not difficult to determine that the saros is fundamental period. Variants of the calculations performed for different values of the errors of the phases $\Delta\varphi = 0.05$, $\Delta\varphi = 0.08$ (fig. 6) and $\Delta\varphi = 0.14$ show the stability of the result. Application of the modified algorithm to the canon of solar eclipses shows that saros can not be found by observing of the partial eclipses. This result is quite expected, since the value of the phase during a solar eclipse depends not only on the restoration of latitude, but also on the location of the observation. In the case of lunar eclipses there is no such dependence, so the method of selection eclipse phase is effective.

As a result, it is shown that saros can be found by observing partial lunar eclipses. However, the above case is an idealization, since in practice it is impossible to register every eclipse over a long period. We distinguish two independent factors, which may lead to loss of observations. The observer can skip eclipses with a small phase due to their more difficult visibility and shorter duration. In addition, there is a factor the weather, which reduces the probability of observing the eclipse, regardless of its phase. To account for these factors we have to make a model of loss of the observations.

Simulation of the losing of observations. We assume that the probability of registration ρ of the eclipse with the phase $\varphi \leq 0.5$ equal to the magnitude of phase, $\rho = \varphi$. In the case of $\varphi > 0.5$ we accept $\rho = 1.0$. This means that the eclipse with the phase of $\varphi = 0.10$ placed into the canon with a probability of 10%, the eclipse with phase $\varphi = 0.50$ with a probability of 50%, etc. For larger values of the phases of eclipses, this probability is equal to 100%. We assume that the average annual probability of realization of favorable weather conditions for observing the eclipse of $\rho_{obs} = 0.8$, which corresponds to 292 days. This estimate is quite realistic for Egypt, Greece and Mesopotamia.

By applying n times both probabilities to the canon of eclipses, we obtain the realizations of the canon, which takes into account the loss of observations. For each realization of the canon, we compute the frequency of periods, and after averaging, we obtain a set of mid-frequency repetition periods of eclipses, fig. 8.

Conclusion. Our estimates show that during the observation period of $T = 300$ years about ~ 70 partial lunar eclipses can be registered for this model parameters. This number of eclipses is sufficient to detect the saros, the double saros and the exeligmos among the other periods. Reducing the time of observations in 2 times, will lead to a twofold decrease in the frequency and complicates the allocation of the required periods. Therefore, we can assume that the minimum period of observations of lunar eclipses for the detection of the saros is $200 \div 300$ years at an average density of observations at least ~ 25 (300 years) and ~ 35 (200 years) of partial eclipses per century.

5 Determination of the duration of the lunar months

Beside the challenge to detecting the saros, there is task of determination the duration of the lunar months and "the saros relation" that was reported by Ptolemy: 1 saros = 223 synodic months = 239 anomalistic months = 241 sidereal month = 242 draconic month.

Determination of the duration of synodic month. The duration of synodic month is easiest to define. We should take the time interval between a pair of any lunar eclipses and divide it by the number of passed lunar months. From the analysis of the "Almagest" lunar eclipses it is known that the average error in determining the time of the eclipse is about 30 minutes. When the times of pair of eclipses are subtracted, the resulting error can be smaller or larger by this value. On average, the error increases in $\sqrt{2}$ times and it is about 42 minutes. At the same, this error of 42 minutes is evenly distributed over the number of revolutions of the Moon, and this would reduce the error of determination of the duration of the synodic month. For example, let's take the interval between eclipses $T = 300$ years, which contains approximately 3711 full moons. Then the error of synodic month is ~ 0.7 seconds or $8 \cdot 10^{-6}$ days. The duration of synodic month defined by the pair of eclipses is 29.530594^d, which differs from the exact value on the $6 \cdot 10^{-6}$ days. If we divide 6585.33^d at 29.530594^d, we find that the saros contains 223 synodic months.

Determination of the duration of sidereal month. The method of finding the length of sidereal (star) month is reviewed in detail by Bronshten [BRON]. Over one sidereal period T_{sid} the Moon makes a complete revolution on its orbit relative to the stars. Over one synodic period T_{syn} the Moon makes a return to the Sun (or phase). Since the Sun moves relative to the stars in the ecliptic, it shifts during the sidereal month by some distance. The Moon needs more time to overcome this distance, and therefore, the length of the synodic period T_{syn} is greater than the sidereal period T_{sid}. Thus, the sidereal month characterizes the motion of the Moon, and synodic month characterizes mutual motion of the Moon and the Sun. Therefore, the length of synodic and sidereal lunar months is associated with the duration of sidereal year T. This means that knowing any two periods, the third period can be calculated. Over the one lunation the Sun passes T_{syn}/T part of the ecliptic. Over the one sidereal month, the Moon makes one complete turn relative to the stars, but in order to catch up the Sun, it would need to overcame extra distance $(T_{syn} - T_{sid})/T_{sid}$. Equating these values, we can express T_{sid}: $T_{sid} = T \cdot T_{syn}/(T + T_{syn})$. Using the known value of synodic months, and different estimating the duration of the year [WAER] and [PTOL], we define the corresponding duration of the sidereal month, Table 2.

Author	T, **(source)**	T, **(days)**	T_{sid}, **(days)**
System A	$12;22;8$ syn.m.	365.260299	27.321683
System B	$12;22;7;52$ syn.m	365.259468	27.321678
Meton	$365.25 + 1/76^d$	365.263158	27.321699
Kalipos	365.25^d	365.250000	27.321625
Hipparchus	$365.25 - 1/300^d$	365.246667	27.321607

Table 2. Determination of the duration of sidereal lunar month for different estimates of duration of the year.

Note, that the durations of the sidereal months differ from each other only after the fourth decimal place for any length of the year. It follows that one saros contains 241 sidereal month.

Determination of the duration of draconic month. Let's again use the frequency analysis, but on this occasion we will select the eclipses not only by magnitude and phase, but the position of the Moon relatively of a node. In order that the period may contain a whole number of revolutions, it is necessary that both of the pair of the eclipses occurred near the descending node, or ascending node of the orbit. So the description of the eclipse should be conveyed to the information which part of the lunar disk (northern or southern) was covered by the shadow. Table 3 shows six of the shortest periods with the value of frequency $f \geq 7$.

N	**Period**	**Frequency**	**Synodic month**
1	1387.7	11	47
2	2775.3	12	94
3	3809.6	7	129
4	5197.3	10	176
5	6585.3	23	223
6	7973.3	9	270

Table 3. Six shortest periods with a biggest value of the frequency.

In the column No.2 the lengths of the periods were averaged over the number of frequencies in the period. The ancients did not know the operation of averaging, but they could use the value which is known as mode in modern statistics. To do so, they should round the periods and select the most frequently occurring value. In addition, the accuracy of the determination of the period is affected by the error of fixing the time of the eclipse, which is estimate as about half an hour. Therefore, to simulate the data obtained by ancient astronomers, we have to round the exact times of maximum phases, which we use in the calculation, to a half hour. The column number 3 shows the frequency of the repetition period, and the column No.4 indicates the number of synodic months, which corresponds to the period. We can find that the periods can be expressed in terms of each other with a good precision: $P_2 = 2 \cdot P_1$, $P_4 = P_3 + P_1$, $P_5 = P_3 + 2 \cdot P_1$, $P_6 = P_3 + 3 \cdot P_1$. So let's take periods P_1 and P_3 for the future analysis.

Let us estimate the duration of the draconic month. The relation $T_{syn} > T_{dra}$ can be found through the observation of the eclipses. To do this we have to consider several eclipses in a row, which occurred near the same node. It is not difficult to find that point at which the eclipse took place, moving against the direction of movement of the Sun on the ecliptic. Consequently, the longitudes of points in which the eclipses occur, are decreasing. This is possible only when: $T_{syn} > T_{dra}$.

Further, we use another well-known observational data that the minimum time between eclipses τ is usually 6 months, $\tau = 6$. Assume that in some eclipse, the Moon has passed exactly in the node. At this time the distance of the Moon to the opposite node is $29.53/2 = 14.77^d$. To simplify the evaluation, we assume that the Moon moves in orbit uniformly. In the next full moon, the node in which a month earlier there was an eclipse, will overtake the Moon on $T_{syn} - T_{dra} = 29.53 - T_{dra} = x$ days, fig. 9. Accordingly, each month, the opposite node will approach to the Moon on x days and it will catch up the Moon through $14.77/x = 6$ month. Hence it follows $x = 2.46^d$ and $T_{dra} = 27.07^d$. The estimate of T_{dra} is inaccurate, because we did not take into account the size of the Earth's shadow and we assumed that the orbital motion of the Moon is uniform. Suppose that our assumptions led to the error $\pm 1^d$. Then the length of draconic month ranges from 26.07^d to 28.07^d.

We can find the duration of the draconic month by using the condition that each of the period P_1 and P_3 contains an integer draconic months. Considering the periods P_1 and P_3 we obtain a system of two equations of the form $n_i \cdot T_{dra} = P_i$, which contains three unknown variables. In general case, this equation can not be definitely solved, but in our case the numbers n_i are integer and, moreover, there is a estimate for $26.07^d \leq T_{dra} \leq 28.07^d$. This allows counting on the possibility of solving the system of equations.

Let 's take the first period $P_1 = 1387.7$ and divide it by estimating the length of draconic month $P_1/T_{dra}^{min} = 1387.7/26.07 = 53.22 \approx 54$ and $P_1/T_{dra}^{Max} = 1387.7/28.07 = 49.4 \approx 49$. This means that period may contain from 49 to 54 draconic months. From similar reasoning, we can find that the period P_3 contains from 135 to 147 draconic months. Dividing the value P_1 by the series of the natural numbers from 49 to 54, we get 6 different variants of the duration of the draconic month. Next, using similar method, we will calculate the 13 possible variants of duration of draconic month, which are corresponding to period P_3, table 4. The coinciding numbers in two columns are corresponding to the duration of the draconic month.

n_1	P_1=**1387.7**	n_3	P_3=**3809.6**
49	28.3193	135	28.2185
50	27.7530	136	28.0110
51	*27.2088*	137	27.8066
52	26.6856	138	27.6051
53	26.1821	139	27.4065
54	25.6972	140	*27.2107*
		141	27.0177
		142	26.8274
		143	26.6398
		144	26.4548
		145	26.2724
		146	26.0925
		147	25.9150

Table 4. Determination of the duration of the draconic month.

Formally, periods $T_{dra} \sim 27.21$ fit best the criteria of search, which corresponds to $n_1 = 51$ and $n_3 = 140$. Let's consider a less precise periods $T_{dra} = 27.75 \div 27.81$ ($n_1 = 50$, $n_3 = 137$) and $T_{dra} = 26.64 \div 26.69$ ($n_1 = 52$, $n_3 = 143$), which will be considered as an alternative variants. In order to final selection, we will divide the greatest period $P_6 = 7973.3$ by the estimated duration of the lunar months, Table 5.

$P_6 = 7973.3$	**m**
27.21	293.03
27.75	287.33
27.81	286.71
26.64	299.30
26.68	298.85

Table 5. Determination of the duration of the draconic month.

The dividing of period P_6 leads to fractional balances for any cases except the first case. Therefore, the required period is $T_{dra} = 27.21^d$. Dividing saros on T_{dra} we obtain the relation 223 synodic months = 242 draconic month.

Determination of the duration of anomalistic month. Let us turn to the observations. Near perigee, the Moon has the highest speed and a maximum angular size. Vice versa, near apogee, the Moon is moving with the lowest speed and it has a minimum visible angular size. The ancient observations with the measured diameter of lunar disk are unknown. Furthermore, even such observations somewhere exist, they are unlikely to have sufficient accuracy.

According to Ptolemy, to clarify the length of the anomalistic month Hipparchus used eclipses with the similar magnitude (i.e. phase) and their duration. However, let's note that the Hipparchus solved the problem of refinement and the duration of the anomalistic month was previously known. Following the idea of Hipparchus, we supplement the modified frequency method with an additional condition on the duration of eclipses. The duration of a pair of compared eclipses should differ by less than 20 minutes, $\Delta t \leq 20^m$. Let's compare the new values of frequencies with the previous calculation, which corresponds to $\Delta\varphi = 0.11$ and any value Δt, fig. 7. Obviously, with the appearance of conditions at Δt, the frequency of those periods, which contain the integer (or nearly integer) number of anomalistic months do not change. The comparison of frequencies values shows, that saros and exeligmos contains an integer anomalistic months. Perhaps the estimate of an error $\Delta t \leq 20^m$ is unachievable for the time of Hipparchus and we have to use assessment $\Delta t \sim 30^m$ and worse. In this case, the number of periods, which contain the multiple number of the anomalistic month, will increase. Note that the position of the Moon relative to its apogee does not affect the possibility of occurrence of the eclipse. Therefore, in this case we can not apply an algorithm that was used to determination for the duration of the draconic month.

Let us turn to Ptolemy, again. He asserts that the Moon can have the maximum and minimum speed at any sign of the zodiac. Consequently, the measurements of the Moon speed existed in that times. It is possible to measure, that the speed of the Moon at apogee is about 12 degrees per day or 30 arc minutes per hour, and the speed of the Moon at perigee is about 15 degrees per day or $37.5'$ per hour. If we compare the angular displacement for 1 hour, then the difference will be about $7.5'$. This value is less than the characteristic measurement error of $10'$. However, near the full moon it is possible to observe the motion of the Moon during at least 8 hours. In this case, the difference of angular displacement at perigee and apogee will be about $55'$, fig. 10.

From this figure follows that more determination of the position in the apogee is more accurate, as the error value of $10'$ corresponds to 6 points (days) at apogee against to 8 points at perigee. In addition, the position of the apogee can not be accurately determined by one or two observations. However, if the motion of the Moon was observed near the apogee for several days, it is easy to determine the position of the apogee relative to the fixed stars up to a day. Since the speed of angular motion of the Moon at apogee is about 12 degrees per day, the position of apogee may be determined with an accuracy of 12 degrees. After one revolution, we can determine apogee position with the same accuracy.

Then, the resultant error of a pair of observations will be in $\sqrt{2}$ times more and will be about 17 degrees. Let's coarsen this value to 20 degrees. Then, instead of the true value of the rotation period of apogee $t_a = 8.85$ years, we obtain a possible range of its value from 8.36 to 9.37 years. From these observations it is possible to determine the direction of the movement of the apogee relative to the fixed stars. It turns out that the apogee is moving forward on the ecliptic, hence, the duration of anomalistic month exceeds the duration of the synodic month: $T_a > T_{sid}$. Now, let's repeat the reasoning used to determine the duration of the sidereal month. Let the apogee shifts to $T_a > t_a$ on revolution during one anomalistic month. During this time, the Moon makes one rotation relative to the fixed stars, but in order to catch up the perigee it need to pass extra distance $T_a - T_{sid})/T_{sid}$. Equating these values we have for T_a: $T_a = t_a \cdot T_{sid}/(t_a - T_{sid})$. Substituting the edge points of the interval $T_a \in [8.36; 9.37]$, we obtain a possible range of estimates $T_a = 27.541 \div 27.658^d$. Received interval is well corresponding to the exact value of the anomalistic month $T_a = 27.5545^d$. This result can be ascertained by using of the saros, which contains an integer number of anomalistic month. If we divide the estimated length of the saros on the duration of the anomalistic month we get ratings: $6585.3/27.541 = 239.11$ and $6585.3/27.658 = 238.76$. The nearest integer number of these two numbers will be 239, therefore, saros contains 239 anomalistic months. Consequently, the refined duration of the anomalistic month is $T_a = 6585.33/239 = 27.5537^d$.

6. Conclusion

In this paper, we propose an experimental method for detecting saros based on observing eclipses. It was shown, that the saros could be detected by analyzing the repetition periods of partial lunar eclipses with the same phase. This requires regular observations of eclipses over $200 \div 300$ years at an average density of at least $\rho \sim 30$ observations per a century. The observer has to register the time, phase of eclipse, and the position of the Earth shadow on the lunar disk.

The experimental method of determination of the duration of the draconic and anomalistic month was offered. The knowledge of these values allows to get the "saros relation", which Ptolemy assigned to the Babylonian astronomy. The determination of the duration of the anomalistic month requires the astrolabe or another instrument, which allows to measure the motion of the Moon and time.

It is shown that exeligmos can be found experimentally trough the observation of the solar eclipses.

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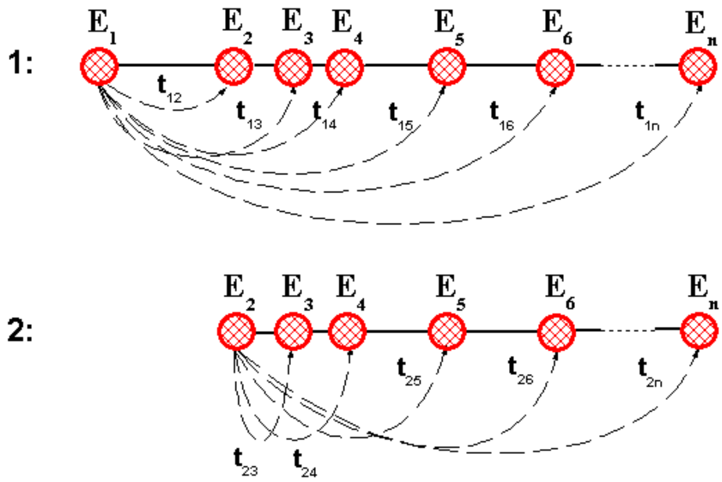

Fig. 1. The algorithm of searching of eclipse repetition periods.

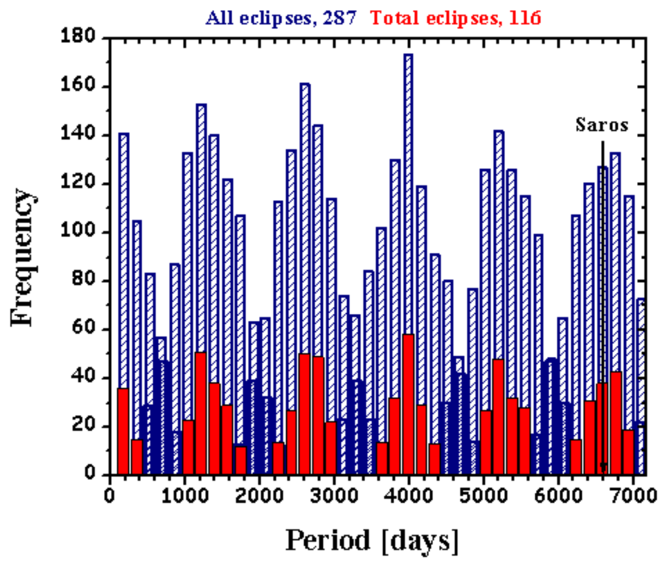

Fig. 2. Short time periods of recurrence of lunar eclipses calculated from the model data. Hatched dark blue columns correspond to the variant of calculation, where all lunar eclipses were used. Red columns correspond to the variant, in which frequencies is determined only by a total eclipses.

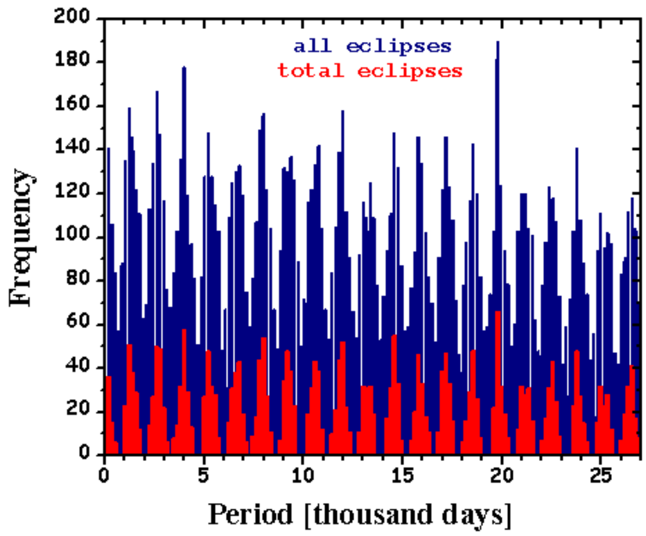

Fig. 3. Long time periods of recurrence of lunar eclipses calculated from the model data. Dark blue columns correspond to the variant of calculation, where all lunar eclipses were used. Red columns correspond to the variant, in which frequencies is determined only by a total eclipses.

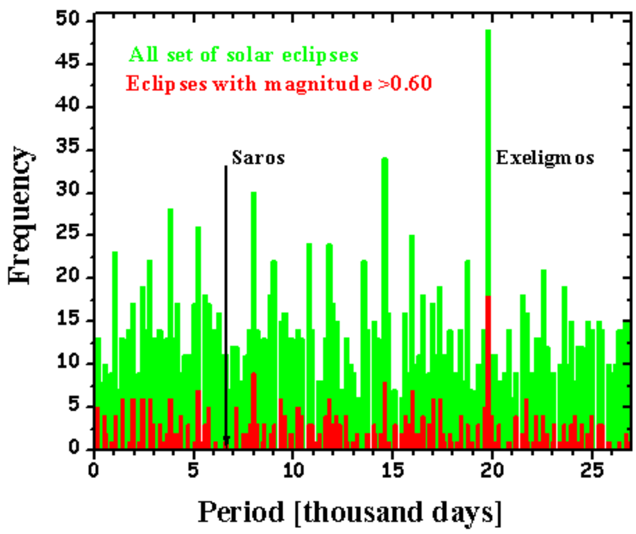

Fig. 4. Periods of a repetition of solar eclipses, calculated from model data. The red color is denoted the variant where the phase of the eclipse exceed 0.6.

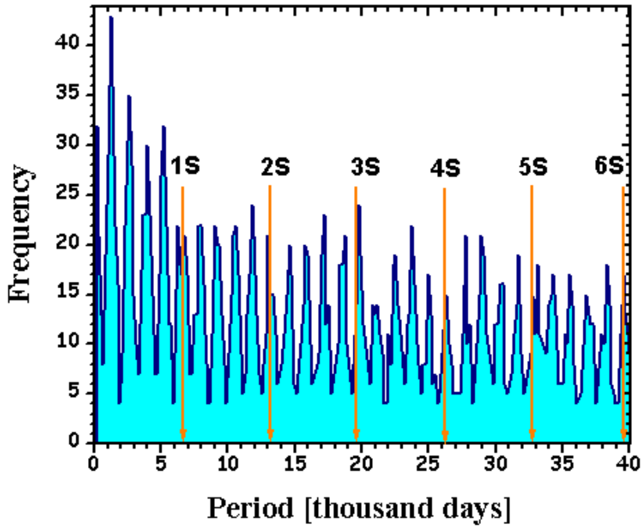

Fig. 5. Periods of recurrence of eclipses, calculated by 154 lunar eclipses from the Chinese chronicles

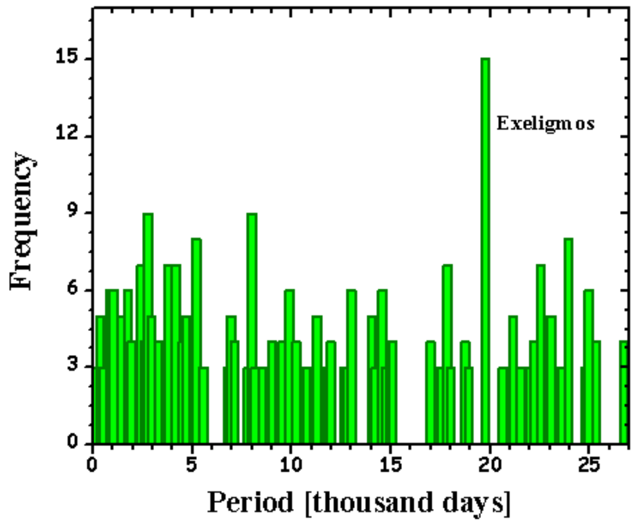

Fig. 6. Periods of recurrence of eclipses, calculated by 95 solar eclipses from the Chinese chronicles.

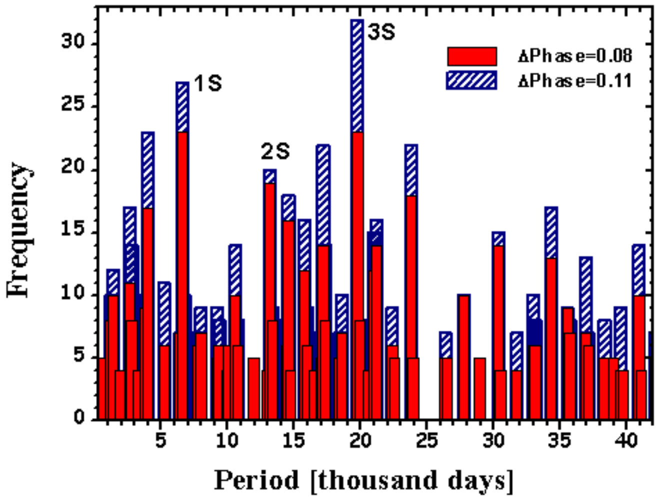

Fig. 7. Periods of repetition of the partial lunar eclipses with the same phase calculated by the model canon. Dark blue color corresponds to the condition, $\Delta\varphi = 0.11$ and the red color corresponds to the additional condition $\Delta\varphi = 0.08$.

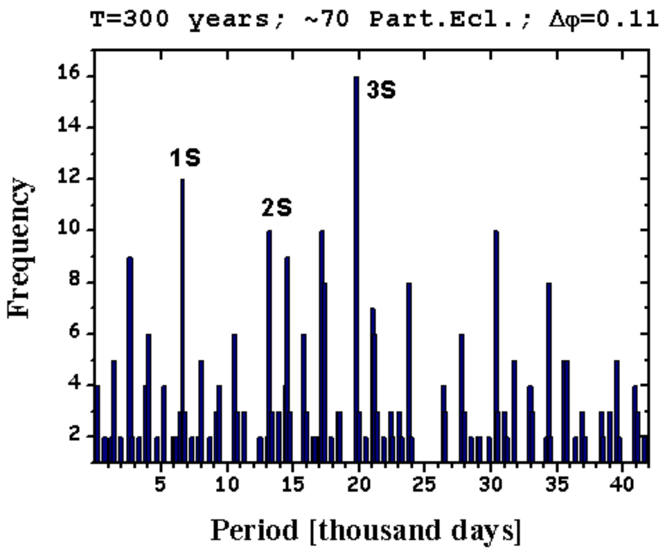

Fig. 8. Periods of repetitions of the partial lunar eclipses with the same phase calculated by the model canon. The calculation takes into account the probability of detection of eclipses and the factors of weather conditions.

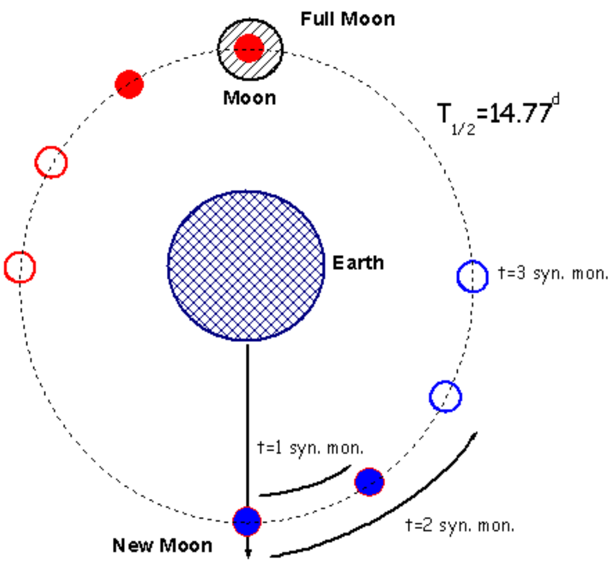

Fig. 9. Figure illustrating the method for estimating the duration of draconic month. The blue dashed circle denotes the Earth, the black circle - Earth's shadow. At the initial moment the "red" node is in the Earth's shadow, and "blue" node is on the line of the Earth - the Sun. With every passing month, the "blue" node approaches to the cone of the Earth's shadow, and "red" node removes away. Approximately, through 6 months, the "blue" node enters the Earth's shadow and an eclipse will take place.

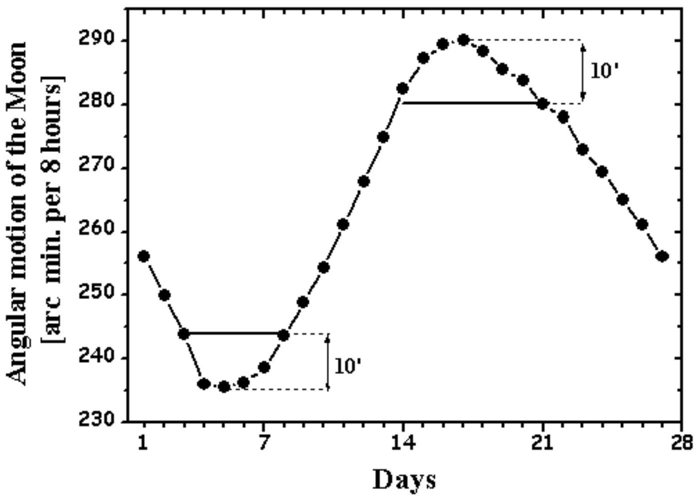

Fig. 10. The visible movement of the Mood during 8 hours. Near apogee and perigee, the character measurement error of 10′ is marked.