Model of the stellar visibility during twilight

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Abstract. On the basis of observational data a model of stellar visibility during twilight was developed. The proposed model can predict the moments of beginning and ending of stellar visibility. Theoretical estimates, defined on the basis of the model, are in good agreement with the observational data of Ptolemy, Schoch and Schaefer.
Key words: stellar visibility during twilight, heliacal risings and settings, history of astronomy

Модел на звездната видимост в полумрак

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На базата на наблюдателни данни е построен модел на звездната видимост по време на полумрак. Предложеният модел може да предскаже моментите на начало и край на звездна видимост. Теоретичните оценки, базирани на модела, са в добро съгласие с наблюдателните данни на Птоломей, Шох и Шефер.

1 Introduction

The problem of theoretical determination of the moments of stellar visibility in twilight is of interest for historical-astronomical investigations. The numerical calculation allows simulating the conditions of planets and stellar visibility, depending on the conditions of observation and atmospheric conditions. It makes possible to solve a wide class of problems in the field of history of astronomy.

For the first time the problem of determining the conditions of heliacal risings and settings of the planets was formulated and solved by C. Ptolemy's in the 13^{th} book Almagest [Toomer, 1998]. From the observations, Ptolemy calculated the so-called "arcus visionis" for Mercury, Venus, Mars, Jupiter and Saturn. To simplify, we will use the term "arc of visibility" instead of "arcus visionis".

Arc of visibility γ is the sum of the altitude of a star H and the solar depression below the horizon h at the time of observation: $\gamma = H + h$. In addition let's note, the arc of visibility means a set of parameters H and h which correspond to the minimum possible value of γ [Purrington, 1988], Fig.1. The magnitude of the arc of visibility depends on the brightness of the star, its spectrum, the difference of azimuths between the Sun and the star $\mid \Delta\theta \mid$, and transparency of the atmosphere, which is characterized by the extinction coefficient k. When the Sun goes down below the horizon the sky brightness decreases, and more faint stars become visible. On the other hand, a star follows the Sun toward the horizon, which leads to an increased atmospheric absorption in the direction to the star. It brings to decreasing visible brightness of the star and to deterioration of the observation conditions. (During sunrise the opposite situation is realized.) Thus, a stellar visibility is defined by two inverse processes. At the beginning, the star appears in the twilight after sunset, and then it disappears at some distance from the horizon. The set of values H and h that defines the arc of visibility γ is achieved when the time of stellar visibility during twilight is minimal.

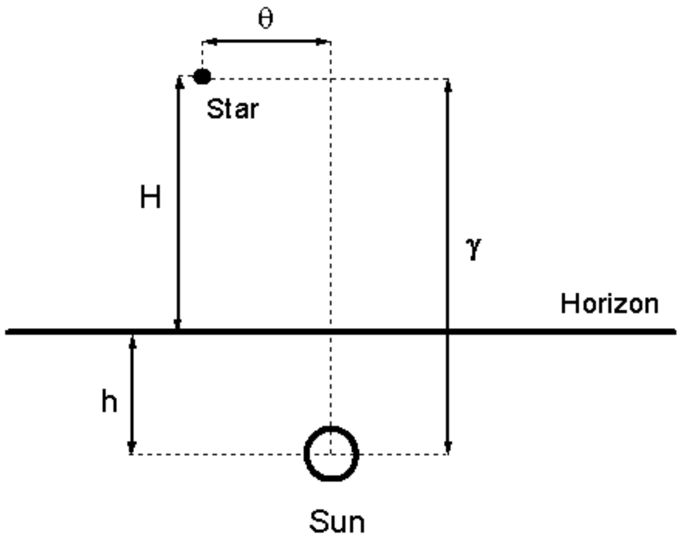

Fig. 1. *Determination of arc of visibility γ. H - stellar altitude, h - solar depression below the horizon, θ - difference of the stellar and solar azimuths.*

Let's note one important detail. In general case $H(t)+h(t) \neq const$, that is especially noticeable when $\mid \Delta\theta \mid > 30^0$. For example, suppose that at the initial moment the Sun is exactly on the mathematical horizon, a star at a certain altitude H. After a while, the Sun will have depression η, but the altitude of star will differ from the value $H - \eta$. This phenomenon can be easily explained for the case limit of a non-setting star. It is obvious that the Sun will be going down below the horizon from the moment of sunset to the moment of lower culmination, while a non-setting star will be "sliding" along the horizon. So a precise determination of the arc of visibility is possible only by mathematical simulation.

Let's note that Ptolemy defines an arc of visibility of the planet equal to the Sun depression below the horizon at a time when the planet itself is on the horizon [Kurtik, 1998]. If the altitude of the planet is equal zero $H = 0^0$, atmospheric absorption is so great that observer is not able even to register the Venus. However, by Ptolemy, the depression of the Sun is a calculated value. It was determined from the arc which corresponds to a minimum elongation of the planet, when a twilight visibility of the planet is possible. The definition of the arc of visibility by Ptolemy can be reduced to the determination of the arc of visibility by Parrington. Observing from

Alexandria, Ptolemy specially carried out observations when the Sun and the planet were located near the summer solstice. For this case the condition $\gamma = H(t) + h(t) \approx const$ is true. Let's assume that at the first moment t_0, the planet becomes visible in twilight at an altitude $H \approx 3^0 \div 6^0$. At the time t_1, planet riches horizon, when it's altitude is equal zero to $H = 0^0$. Then a depression of the Sun at moment t_1 is equal arc of visibility at the time t_0: $\gamma(t_0) \approx \gamma(t_1) = h(t_1)$.

By the late 19^{th} and early 20^{th} centuries many cuneiform tablets with various astronomical content had been found in Mesopotamia. In the so-called "diaries" the dates of heliacal rising and setting of the planets are given. Since the dates of observation are known, it allows us to define the arc of visibility for each observation. The text MUL.APIN contains observations of heliacal risings of bright stars, whence we can get estimates for the values of arcs of stellar visibility. On the basis of Babylonian data, Schoch [1924] constructed a linear model $\gamma(m)$ which allows to calculate the arc of visibility for heliacal rises and settings. He estimates that model is applicable when $\mid \Delta\theta \mid \leq 25^0$.

Among modern studies the greatest number of works devoted to the definition of the visibility of the heavenly bodies belongs to Schaefer. In paper [Schaefer, 1987], the author set the task to develop a model based on "astronomy and physiology," which would describe the visibility of stars without using Ptolemy's concept of the arc of visibility. The main idea of Schafer's approach is that he calculates the brightness of the sky background in the direction of a star at a time of observation. The sky brightness is recalculated into stellar magnitudes and then it is compared with the brilliance of a star. As result, the possibility of stellar visibility is determined.

The main difficulty of Schaefer's model is that the distribution function values $\Delta_{Pred} - \Delta_{Obs}$ differ significantly from the Gaussian distribution (see Fig.1), and it's mathematical expectation is $M(\Delta_{Pred} - \Delta_{Obs}) \approx 5 \div 6$. This is evidence of the presence of systematic errors. It should be noted, that in this assessment values $\mid \Delta_{Pred} - \Delta_{Obs} \mid > 10^d$ were not taken into account although they exist,as can be seen from the content of table 1, p. 20. Therefore, the model has greater errors. Schaefer himself explains occurrence of such errors by the presence of barely visible clouds near the horizon, which alter the extinction coefficient during the observation. The extinction coefficient is really likely to change, but it can be measured during setting of stars in the vicinity of the horizon. In this case, the error of determination of extinction coefficient will be minimal. On the other hand, in table 1 faint stars (ι Psc) are present, their visibility is hardly possible at altitudes less than $H \sim 6^0$. For the majority of faint stars, the altitude above the horizon at which the star can be observed would be even greater. With increasing altitude H the value of total atmospheric absorption rapidly decreases. In this situation, the influence of extinction coefficient (and errors of it determination) on stellar visibility also decreases. Therefore, at least for some cases the dates mismatch $\Delta_{Pred} - \Delta_{Obs}$ is reasonable to be explained rather by the inaccuracy of the model than by the error of measuring the extinction coefficient.

The final study [Schaefer, 1993] contains a great number of theoretical and background material, however, when the problem of twilight stellar visibility is discussed, the author refers to the previous works [Schaefer 1985], [1987]. It means that new results on this issue are absent.

Our aim is to develop a model that will be able to calculate a possibility of stellar visibility for any arbitrary case.

2 The method of observation

To solve the raised problem, we had observed the appearance of stars during twilight for three years and received more than 600 separate observations. The bulk of the observations were conducted in Moscow, where it is rather problematical to observe stars with magnitude less than 3.5^m confidently, due to the strong illumination of the sky. In this regard, we observed the brightest stars, which luminosity exceeds the background of twilight sky.

The aim was to determine the moment of appearance (for evening observations) or disappearance (for the morning observations) of the stars observed by the naked eye. Let's note that evening and morning observations are not equivalent. In the first case, the appearance of the star is expected in some neighborhood and, first of all, observer has to find the star. In the second case, the position of the star is known exactly and we need to fix the time of it's disappearance. This is a simpler task, so the morning observations should be more accurate. In order to remedy for this situation, for evening observations, we marked the azimuth and the height at which appearance of a star was expected. In addition, the stellar position was determined by using the binoculars shortly before the beginning of visibility.

The estimates have shown that this procedure allowed to minimize the difference between evening and morning observations. In one session we watched the appearance of several stars with different brightness. In the first experiments time of the stellar appearance was rounded to one minute. In later observations we fixed time of appearance of the first star and the moments of emergence of all of the following stars were measured by stopwatch relative to the first star.

3 The results of observations

Development of a simple model

Knowing the date, the exact time of observation and the geographic coordinates, we calculated the depression of the solar disk below the horizon and determined the angle between the star and the Sun. Figure 2 shows the extreme conditions when Vega becomes visible. In this set we selected observations where the altitude of the star was greater than $H > 40^0$. Fig. 2, the abscissa is the angular distance Vega from the Sun, which is minimal near the winter solstice ($\sim 62^0$) and maximal around the summer solstice ($\sim 120^0$). The ordinate is the depression of the Sun below the horizon at the moment of appearance of the star. From this figure one can draw several important conclusions.

Firstly, the minimum depression of the Sun below the horizon, when the visibility of Vega is possible is $h_{lim} = -2.5^0$.

Secondly, according to winter, autumn ($x \sim 90^0$) and summer observations, this value can be realized during a year. It means that the value h_{lim} does not depend on the distance between the star and the Sun in the range $60^0 < x < 120^0$. If such dependence exists, it cannot be found due to errors. Similar observations of Altair show that this range can be extended at least up to $60^0 < x < 150^0$. Therefore, as an initial approximation, we assume that the increase in the brightness of the twilight sky begins to exert the influence on the visibility of stars at $x > 60^0$.

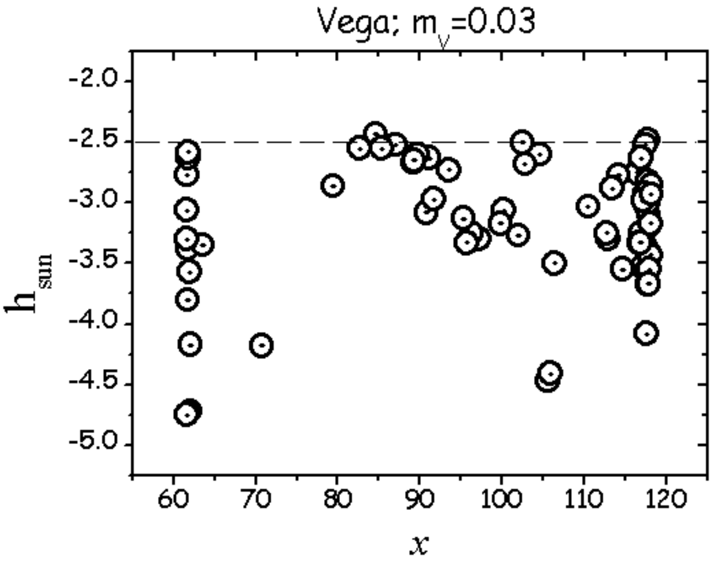

Fig. 2. *Extremely conditions when Vega becomes visible.*

Thirdly, according to the data, summer, autumn and winter observations have different value for the variance of h_{lim}. The variance of h_{lim} is determined by two random processes, they are implementation of the extinction coefficient and the probability of registration of the star. The minimum value of atmospheric extinction is determined by Rayleigh scattering. The scattering of dust and aerosols is added to this value. Among all aerosols the greatest influence on the extinction coefficient has a water vapor. Therefore, the variation of atmospheric absorption is associated primarily with the concentration of water vapor, which is subject to local and seasonal changes.

Thus, the implementation of the minimal value of the extinction coefficient is a random process. According to our estimates, the average error associated with the probability of stellar registration is about $\Delta t \approx 2$ minutes, that corresponds to an error of the value of the solar depression below horizon about $\Delta h \approx 20'$. It means that the extinction coefficient has the greatest influence on the variance of $D(h_{lim})$. According to our observations, the highest probability of realization of clean atmosphere is achieved in the autumn, when the variance $D(h_{lim})$ reaches a minimum value.

Finally, since we have determined the value of h_{lim} for Vega, this star can be used as a star-standard for assessing the transparency of the atmosphere. If during observation we have found that Vega appeared at limiting value of $h_{lim} = -2.5^0 \div -2.75^0$, it means that a pure state of the atmosphere is realized.

Similarly, the limiting values of solar depression h^* were obtained for the other stellar magnitudes. For observation, we selected stars by the following criteria. Firstly, to exclude the effect of increasing the brightness of the sky background at the approach to the sun, the distance between the star and Sun has to exceed 60^0. Secondly, the brightness of the star should not be significantly weakened by the atmosphere, so we used the stars for which the height above the horizon at the time of observation satisfies the condition $H > 40^0$. The observation results are shown in Fig. 3.

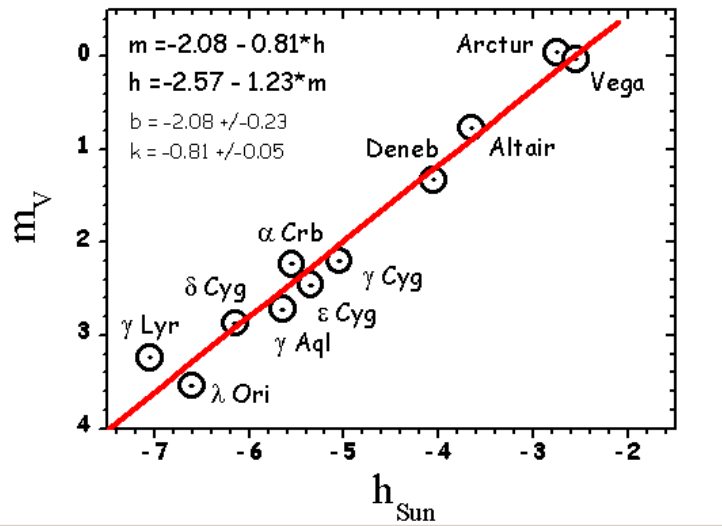

Fig. 3. *The dependence of the extremely visible stellar magnitude m_V on the depression of solar disk below the horizon h_S is given, according to Moscow's observations.*

As a result of observing 11 stars in the range of brightness from -0.04^m to 3.3^m it was established that the dependence of the extremely apparent magnitude m on solar depression below the horizon h is linear:

$$m(h) = -2.08 - 0.81 \cdot h \qquad (1a)$$

In this case, both variables are equivalent. That is, on the one hand, knowing the value of solar depression, we can define a value of extremely apparent magnitude. On the other hand, the registration of a star with known magnitude allows to determine the solar depression below the horizon at the moment:

$$h(m) = -2.57 - 1.23 \cdot m \qquad (1b)$$

This result has two drawbacks. Firstly, it is based on observations of stars up to the third magnitude, so for fainter stars the dependence is unknown. Secondly, the slope of the regression depends on the individual characteristics of the observer. However, we do not know whether the eyesight of the observer is typical of population or not.

In order to address these issues in the summer of 2010 amateur astronomers R.O. Belokrylov and S.V. Belokrylov conducted a series of observations of stars in the village of Zolotarevka in Penza region. The main object was to define the dependence $m(h)$ for stars fainter than the third magnitude and compare the results obtained by different observers.

To observe the following the stars were chosen: α Boo (-0.04^m), α Lyr (0.03^m), α Aql (0.77^m), α Cyg (1.33^m), α CrB (2.23^m), β CrB (3.68^m), γ CrB (3.84^m), θ CrB (4.14^m), ι CrB (4.99^m), ι Lyr (5.28^m). The first five stars from the list we've seen in Moscow, while the rest stars are not available for a Moscow observer. To eliminate the color effects, we selected the star color-index which satisfies the condition $|B-V| \leq 0.2$. The exceptions are Arcturus ($B-V = 1.23$) and δ CrB ($B-V = 0.80$). We see the color of Arcturus, so a correction for night vision is unnecessary. Observations of the star were obtained simultaneously, so we added it to a set of the above mentioned stars. We think that the addition of this star will rather clarify the dependence than make a mistake because of color differences. From the results of observations it follows that a linear relationship $m(h)$ is true until $h_{sun} \approx -8^0$, which corresponds to $m \approx 4^m$, Fig. 4.

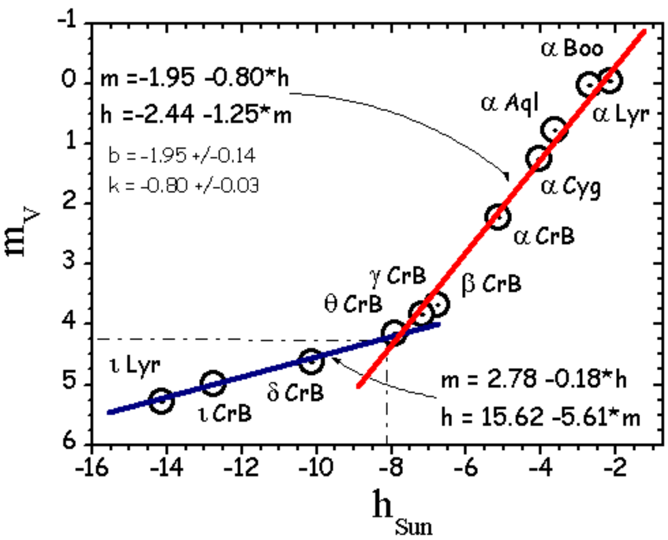

Fig. 4. *The dependence of the extremely visible stellar magnitude m_V on the depression of solar disk below the horizon h_S is given, according to Zolotarevka's observations.*

After passing through this point further depression of the sun reduces the dependence between variables h and m. It should be noted that for the faint stars δ CrB, ι CrB and ι Lyr the accuracy of definition of the moment of stellar visibility is significantly worse than that of bright stars. The average error in determining the moment of visibility of the bright star is about 2 minutes, but for faint stars, it may exceed 10 minutes with the same transparency of the atmosphere. Probably this effect is associated with the approach to the threshold of sensitivity of the human eye, which roughly corresponds to the 6^{th} magnitude.

Observations made in Zolotarevka can be approximated by the two regressions, one of which describes the appearance of bright stars (2a) and (2b), and the other - the faint stars (3a) and (3b):

$$m(h) = -1.95 - 0.80 \cdot h \text{ , where } h > -7.7^0 \qquad (2a)$$
$$h(m) = -2.44 - 1.25 \cdot m \text{ , where } m > 4.2^m \qquad (2b)$$
$$m(h) = -2.78 - 0.18 \cdot h \text{ , where } h \leq -7.7^0 \qquad (3a)$$
$$h(m) = 15.62 - 6.61 \cdot m \text{ , where } m \leq 4.2^m \qquad (3b)$$

The comparison of the two dependencies gives the transition point between the regression $m \approx 4.2^m$ and $h \approx -7.7^0$. The coefficients of regression equations for the brightest stars for Moscow's observations (1a), (1b), and for Zolotarevka's observations (2a) (2b) are identical within experimental error. The coincidence of the slopes means that the observers have about the same sensitivity of the retina. Therefore, we will assume that observers have "standard" eyesight, which is not different from the view of most people. Penza's observations give fewer value of the free term in the regression equation due to a lower value of extinction coefficient and a clean atmosphere. In general, the maximum values of transparency of the atmosphere in Moscow and in Penza are the same. It is because of the geographic coordinates of observation points are about the same, and both cities are in the same climatic zone. Therefore, layers of the atmosphere have similar physical characteristics, which lead to the same values of the extinction coefficient.

Thus, for the brightest stars we use the equation with coefficients, averaged under Moscow and Zolotarevka observations. As a result, the equations describing the appearance of bright stars can be represented by:

$$m(h) = -2.01 - 0.81 \cdot h \text{ , where } h > -7.7^0 \qquad (4a)$$
$$h(m) = -2.47 - 1.23 \cdot m \text{ , where } m > 4.2^m \qquad (4b)$$

For stars with glitter, we will use the equation describing the visibility of faint stars, as determined from observations of Zolotarevka. As a result, the model of the stellar visibility during twilight can be described by a system of two equations.

In constructing the model we used extra atmospheric brilliance of the stars and assumed that all the stars are attenuated by the atmosphere almost equally. An execution of the latter condition was achieved by selection rule: $H > 40^0$. The developed model is applicable for "clean" atmosphere, because we selected the best conditions of visibility of each star on many observations. This state of the atmosphere corresponds to some coefficient of extinction, which have to be determined. To find the value of atmospheric extinction, we observed settings of bright stars of the most pure state of the atmosphere.

We assessed transparency of the atmosphere by using Vega, which was a star-standard. When a setting star was near the horizon, we estimate its magnitude relative to the stars located at an altitude $H > 40^0$ and then we determined extinction coefficient by the Bouguer law. As a result, the observations showed that in Moscow the extinction coefficient for the cleanest air is $k = 0.23 \pm 0.02$ per one atmospheric mass. With sufficient accuracy it can be assumed that, on average, the stars were observed at 1.2 air masses, which corresponds to the average absorption coefficient $k = 0.23 \cdot 1.2 \approx 0.25$.

Consideration of the atmospheric absorption.

To assess the visibility of stars at an altitudes of $H < 40^0$ it is necessary to consider the absorption of the atmosphere. The total attenuation can be expressed as $\Delta m = k \cdot (F(z) - 1)$, where z is the zenith distance, k - extinction coefficient, $F(z)$ - the number of air masses in the direction to the star. Since equation ($4a$) and ($4b$) were obtained for one air mass it should be subtracted from the value $F(z)$. There are many methods for assessing of the value $F(z)$. For example:

$$F(z) = [\cos(z) + 0.025 \cdot e^{-11\cos(z)}]^{-1} \qquad (5)$$

Knowing the value of the light attenuation Δm, we can determine visible stellar magnitude, taking into account atmospheric absorption $m' = m + \Delta m + (k - 0.25)$ and then substitute it in the appropriate equation $h(m) = -2.47 - 1.23 \cdot m$ or $h(m) = 15.62 - 5.61 \cdot m$ (if $m' < 4.2^m$). As a result, we find the limiting value of solar depression h_{lim} necessary for observation of the stars. Term $(k - 0.25)$ is need when the value of extinction coefficient at the time of observation differs from the value for which was determined by relation (1) and (2). This expression can be represented in general form:

$$m' = m + k(F(z) - 1) + (k - 0.25) \qquad (6)$$

In the proposed model of atmospheric absorption we do not consider stellar spectra. That is, we assumed that the blue and red stars are attenuated by the atmosphere equally. In the matter of fact, this is not true because there is an effect of atmospheric reddening and the Forbes effect, that atmospheric absorption is reduced with increasing air mass. The influence of both of these effects can be accounted for by an additional amendment to the formula (6). The necessity of introducing such corrections and assessing of their values will be discussed below in a separate paragraph. So far, we assume that these effects are negligible.

Example No. 1. *Verification of the model. Daytime visibility of Jupiter.*

It follows from the dependence $m(h)$ that at sunrise $h = -(r_{sun} + R) = -0.85^0$ (r_{sun} apparent radius of the Sun, R a value of refraction near the horizon) when the northern edge of the solar disk touches the horizon, we can register stars with magnitude brighter than $m(-0.85) = -1.41^m \approx -1.4^m$. Further extrapolation $h > -0.85^0$ of the dependence is not applicable, since the original model does not describe such conditions. However, we can accept that at a low altitude of the Sun above the horizon we have the marginal estimate, which corresponds to $h = -0.85^0$.

During the observations of 30.06.2009, 01.07.2009 and 20.07.2009 we saw Jupiter, when solar altitude was at the range of $h = 0.9^0 \div 2.5^0$. At the time of observation the planet altitude was $H \approx 21^0$, which corresponds to 2.8 air masses, and its extraatmospheric magnitude was $m_j = -2.6^m$. In these observations Vega ceased to be visible when the values of solar depression were $h(30.06) = -2.48^0$, $h(01.07) = -2.82^0$, $h(20.07) = -2.51^0$. In the first and the third cases observations were made at limit value of solar depression h_{lim} when Vega appeared. It corresponds to the most transparent atmosphere of Moscow. The value $h = -2.82^0$ was obtained from the second observation and it corresponds to limit value of Vega taking into account the error $\Delta h = \pm 20' = \pm 0.33^0$. Consequently, for these observations, we can use the value of extinction coefficient $k = 0.25$. Then, a planet magnitude is: $m' = m_j + k(F(z) - 1) + (k - 0.25) = -2.6 + 0.25 \cdot (2.8 - 1) + (0.25 - 0.25) = -2.6^m + 0.45^m = -2.15^m$

Substituting this value into the equation ($4b$) we obtain a value of solar depression h_{lim}, in which Jupiter can be seen under these conditions: $h_{lim} = -2.47 - 1.23 \cdot (-2.15) = -0.17^0 > -0.85^0 = h$. Consequently, since $h_{lim} > h$ the visibility of the planet in such conditions is possible. Another way, we can get such conclusion by comparing the extraatmosphere magnitude of Jupiter with estimate of magnitude by formula ($4a$): $m' = -2.15^m > -1.4^m = m(-0.85^0)$. Hence, the daytime visibility of Jupiter is possible.

Accounting of the sky background.
In all previous discussions, we used the stars moved away from the Sun at a distance $x > 60^0$. In fact, the regression is based on the observations which satisfy a more stringent condition: $x > 80^0$. At lower values $x < 60^0$ it is necessary to consider the increasing the sky brightness, which grows when the Sun comes near to the star.

To construct a model of the sky background, we used observations of Arcturus and Vega. In summer, Arcturus appears by $1 - 2$ minutes earlier than Vega when both stars are removed from the Sun for more than 80^0. In autumn, the minimal distance between Arcturus and the Sun is about $x \sim 30^0$ and at such distance Arcturus becomes visible 10 minutes later than Vega. Let's define the difference between solar depression in which Vega and Arcturus become visible $\Delta h_{VA} = h_{Vega} - h_{Arctur}$, depending on the distance of Arcturus from the Sun x. We consider separately the behavior of the function $\Delta h_{VA}(x)$ when $x > 60^0$, Fig. 5 and $x < 60^0$, Fig. 6.

When $x > 60^0$ the dependence function $\Delta h^1_{VA}(x)$ on the distance x is absent, so it can be approximated by a constant $\Delta h^1_{VA} = -0.14^0$. In the case $30^0 < x < 60^0$ the dependence $\Delta h^2_{VA}(x)$ is linear: $\Delta h^2_{VA}(x) = 1.82 - 0.0338 \cdot x$. Differentiating the latter equation, we obtain the relation between the differentials: $d(\Delta h^2_{VA}) = -0.0338 \cdot d(x)$. It means that in order to compensate the increasing of the brightness of the sky background at the approach of the

Sun to the star on 1^0 degree it is necessary extra depression of the Sun below the horizon on 0.0338^0 degree.

Let's note that determining the dependence $\Delta h^2_{VA}(x)$ we took into account an amendment to an additional absorption of the light of Arcturus by the atmosphere. This amendment considers the fact that during the closest approach to the Sun, Arcturus visibility occurs at low altitudes $H < 40^0$ where the value of atmospheric absorption begins to exert a stronger influence. In this case, we cannot assume that Arcturus and Vega are attenuated by atmosphere equally. Taking into consideration the amendments we will define an atmospheric absorption Δm for each experiment and substitute it into the differentiated equation (4b): $dh = -1.23 \cdot dm$. After calculating the correction Δh we will adjust the value of $\Delta h^2_{VA}(x)$: $\Delta h^2_{VA} = h_{Vega} - h_{Arctur} + \Delta h$.

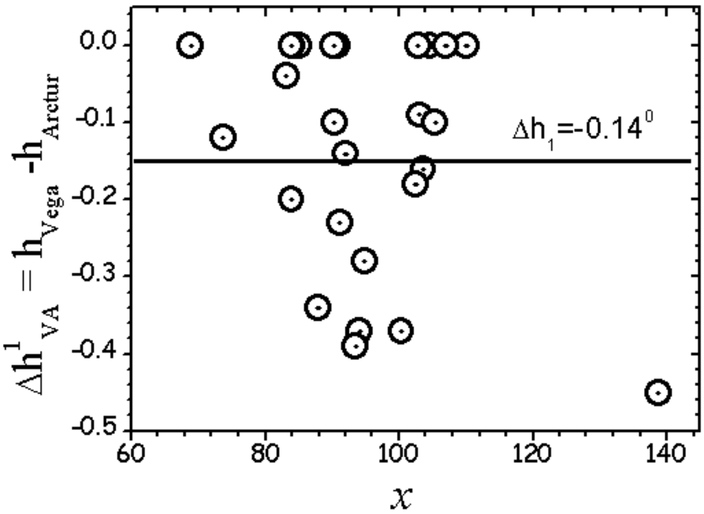

Fig. 5. *Dependence of the function $\Delta h^1_{VA}(x)$ when the distance between a star and the Sun is greater than $x > 60^0$.*

Example of calculation. In the evening on October 10, 2009 we recorded the appearance of Vega at 18^h55^m Moscow time and 19^h02^m appearance of Arcturus. The times of the beginning of stellar visibility correspond to the solar depression $h_V = -2.4^0$, $h_A = -3.4^0$ and the altitude of Arcturus at the time of observation was $H \approx 28^0$. According to observation of Vega the value of solar depression corresponds to the cleanest atmosphere, which is implemented in Moscow. Therefore, we set $k = 0.25$. Then, the amount of additional absorption will be $\Delta m = 0.25(F(62) - 1) = 0.28^m$, and the value of correction will be $\Delta h = -1.23 \cdot 0.38 = -0.34^0$. Finally, taking into account atmospheric absorption we'll get $\Delta h^2_{VA} = h_{Vega} - h_{Arctur} + \Delta h = 0.66^0$, which differs by one-third of the value without correction $\Delta h^2_{VA} = h_{Vega} - h_{Arctur} = 1.0^0$.

Equating the regression equation Δh^1 and Δh^2 we'll define "the return point" η, Fig. 7. The sense of "the return point" lies in the fact that when approaching the Sun more closely than $\eta = 58^0$ it is necessary to consider the increase of the sky brightness. The increasing of the sky brightness can be represented as an additional amendment h^*, which is added to the calculated value of Sun's depression limit h_{lim} when the chosen star becomes visible:

$$h^* = -0.0338 \cdot (58^0 - x), \text{ when } x < 58^0 \qquad (7)$$
$$h^* = 0, \text{ when } x \geq 58^0$$
$$h_{theor} = h_{lim} + h^* \qquad (8)$$

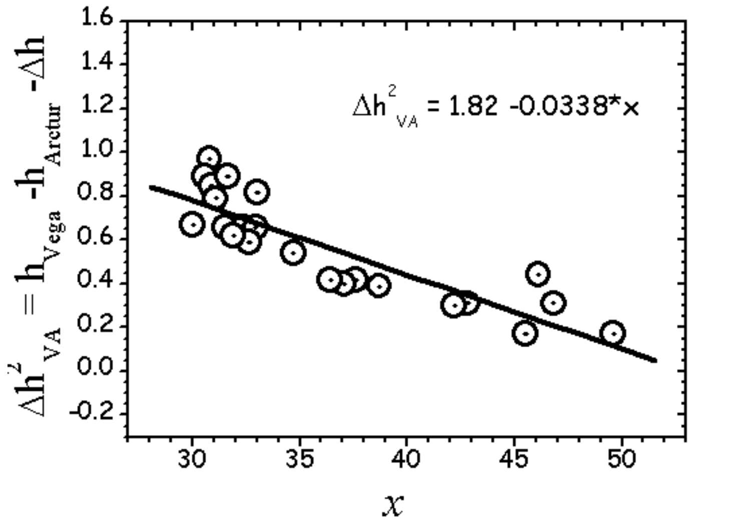

Fig. 6. *Dependence of the function $\Delta h^2_{VA}(x)$ when distance between a star and the Sun is less than $x < 60^0$.*

For constructing a model taking into account the sky background, we used the observations of Arcturus and Vega, although as a comparison one can be used any another star. However, similar functions, based on the observations of Arcturus and γ Cyg, as well as Arcturus and γ Aql have a greater variance of observations, and as a consequence, great errors of the coefficients of equations. This effect is due to the fact that during the observation the transparency of the atmosphere can be changed by the appearance of fine subtle clouds. To reduce the influence of atmospheric conditions the comparison stars have to be selected so that all-stars appear and disappear at about the same time.

The amendment (7) was obtained from observations a pair of stars with roughly the same brightness. However, it does not mean that if we replace the star closest to the Sun (Arcturus) for a star with another magnitude, the equation (7) will remain unchanged. Let's verify this.

Example No.2. *Verification of the model for other stars with another magnitude for a known value of the extinction coefficient.*

On January 11 and 12 2010 Altair $m_V = 0.77^m$ was registered at the altitude $H = 24^0$ and the distance from the Sun $x = 31^0$. According to observations of Vega, the state of atmosphere was close to optimal since $h(Vega; 11.01) = -2.63^0$, and $h(Vega; 12.01) = -2.57^0$. Therefore, we assume that the value of extinction coefficient $k = 0.25$. Then, we calculate a theoretical value h_{theor} and compare it with the value that was obtained from observations h_{obs}.

At the altitude of $H = 24^0$ an additional atmospheric absorption is $\Delta m = k \cdot (F(66) - 1) = 0.25 \cdot (2.46 - 1) = 0.36^m$, then a limit solar depression without increasing of the sky background is equal to $h_{lim} = -2.47 - 1.23 \cdot (0.77 + 0.36) = -3.86^0$. The sky background gives extra amendment $h^* = -0.0338 \cdot (58 - 31) = -0.91^0$, then a theoretical value will be $h_{theor} = h_{lim} + h^* = -4.78^0$. The measured values $h_{obs}(Altair; 11.01) = -4.75^0$ and $h_{obs}(Altair; 12.01) = -5.03^0$ are in the good agreement with the theoretical estimate.

Example No.3. *Verification of the model for other stars with another magnitude for a known value of the extinction coefficient.*

On January 11 and 12 2010 Tarazed (γ Aql) $m_V = 2.72^m$ was registered at the altitude $H = 23^0$ and the distance from the Sun equal to $x = 32^0$. The correction for atmospheric absorption is $\Delta m = k \cdot (F(67) - 1) = 0.25 \cdot (2.56 - 1) = 0.39^m$, whence it follows that a limit solar depression is $h_{lim} = -2.47 - 1.23 \cdot (2.72 + 0.39) = -6.30^0$, and extra amendment on increasing of the sky background is equal $h^* = -0.0338 \cdot (58 - 32) = -0.88^0$. Then a theoretical value is $h_{theor} = h_{lim} + h^* = -7.18^0$. The measured values are $h_{obs}(\gamma \text{ Aql}; 11.01) = -6.70^0$ and $h_{obs}(\gamma \text{ Aql}; 12.01) = -6.97^0$. In the first case, the difference from the theoretical result is about 0.5^0. Since the time of fixing the error is not less than 0.33^0, this result is also quite good.

Thus, according to the examples $No.2$ and $No.3$, the model that describes the increase of sky brightness is independent on a stellar magnitude and it can be successfully applied for stars with different brightness.

Consider the more general case where the extinction coefficient is unknown or it has different values near the zenith and near the horizon. In this case, we can use a pair of observations stars, which are located near horizon close to each other.

Example No. 4. *Verification of the model at the unknown value of the extinction coefficient.*

On June 28 2010 at $23^h 14^m$ Moscow summer time Capella $m_V = 0.08^m$ was registered at the altitude $H = 13^0$ during the minimum distance from the Sun $x = 27^0$. This moment corresponds to the depression of the Sun

$h_{obs}(Capella; 28.06) = -5.67^0$. From these data, we calculate the coefficient of atmospheric absorption. The amendment is $h^* = -0.0338 \cdot (58-27) = -1.05^0$ then $h_{lim} = h - h^* = -4.62^0$. From the equation (4$b$) $h_{lim} = -2.47 - 1.23 \cdot (m_V + \Delta m)$ we express the absorption in the direction of the star $\Delta m = -(h_{lim} + 2.47)/1.23 - m_V = 1.67^m$. Next we find the coefficient of extinction $k = \Delta m / F(77) = 0.38$.

That evening, at $23^h 48^m$ we watched the appearance of β Aur with extraatmosperic magnitude $m_V = 1.90^m$ when $H = 12^0$ and $x = 22^0$. According to (6) the absorption in the direction of the star is $\Delta m = 0.38 \cdot (F(78) - 1) + (0.38 - 0.25) = 1.58^m$, whence it appears $h_{lim} = -2.47 - 1.23 \cdot (1.90 + 1.58) = -6.75^0$ and $h^* = -0.0338 \cdot (58 - 28) = -1.22^0$. Hence the theoretical value of the solar depression is $h_{theor} = -7.97^0$ when the measured value is $h_{obs}(\beta$ Aur$; 28.06) = -7.90^0$.

According to the examples, the constructed model quite well describes the appearance of stars in the twilight, taking into account the atmospheric absorption and non-uniform brightness of the sky at the values $x \geq 22^0$. However, the observations available to us are not enough to test the applicability of the model at $x \leq 22^0$. So we will extrapolate the equation (7) to the case $x \leq 22^0$ and check whether this extrapolation is valid.

Model of stellar visibility during twilight

1. As a result of observation we has been determined the dependence, which connects the limit value of apparent magnitude with a solar depression below the horizon:

$$m(h) = -2.78 - 0.18 \cdot h \text{ , where } h \leq -7.7^0 \qquad (3a)$$
$$h(m) = 15.62 - 6.61 \cdot m \text{ , where } m \leq 4.2^m \qquad (3b)$$
$$m(h) = -2.01 - 0.81 \cdot h \text{ , where } h > -7.7^0 \qquad (4a)$$
$$h(m) = -2.47 - 1.23 \cdot m \text{ , where } m > 4.2^m \qquad (4b)$$

2. These relations allow to estimate the possibility of stellar visibility, for the stars which are located at a distance from the Sun great than $x > 58^0$ and the altitude $H \geq 40^0$. When the altitude is less than $H < 40^0$ it is necessary to introduce an amendment to the atmospheric absorption and re-calculate the star's magnitude.

$$m' = m + k(F(z) - 1) + (k - 0.25) \qquad (6)$$

The recalculated value of stellar magnitude has to be substituted in equations (3b) or (4b) and as result we'll get h_{lim}.

3. When the distance between the star and the Sun is less than $x < 58^0$ it is necessary to consider the background of the sky. In this case to register the star an additional value of solar depression below the horizon is necessary:

$$h^* = -0.0338 \cdot (58^0 - x), \text{ when } x < 58^0 \qquad (7)$$
$$h^* = 0, \text{ when } x \geq 58^0$$

4. Theoretical value of solar depression is necessary to register the star is:

$$h_{theor} = h_{lim} + h^* \qquad (8)$$

4 Verification of the model on other people's observational data.

To test the model using observational data of others, we used the arcs of visibility, which values are defined as follows. Let's assume that, in the initial moment t_0 the Sun has set just below the horizon, and the star with magnitude m and at distance from the Sun x is at an altitude H above the horizon. Having measured some way the extinction coefficient k we can calculate a value h_{theor} and compare it with a value of solar depression $h(t_0)$. Let's assume $h_{theor}(t_0) - h(t_0) < 0$.

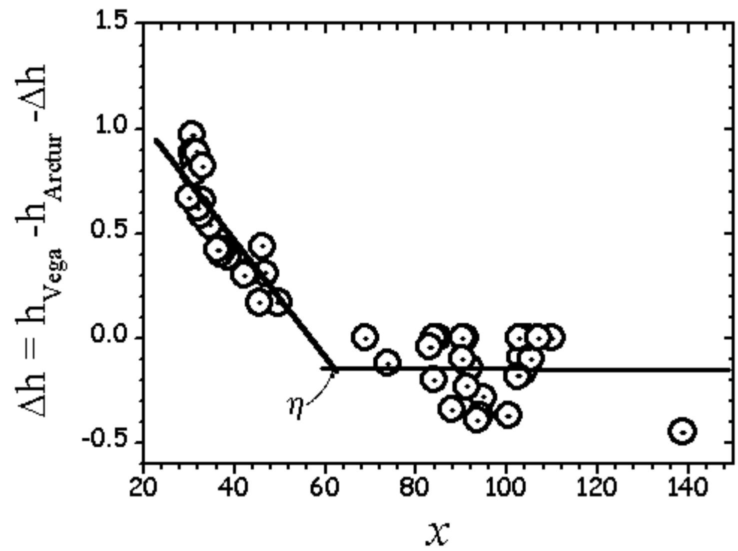

Fig. 7. Determination of the return point η

Eventually, values $h_{theor}(t)$ and $h(t)$ change, and as soon as the difference becomes positive $h_{theor}(t_0) - h(t_0) > 0$, the star begins to be visible in the rays of sunset. At the beginning of stellar visibility t_1, we define the quantity $\Gamma(t_1) = H(t_1)+h(t_1)$. With further depression a star is being grown weakened by the atmosphere and at time t_2 the difference is again negative $h_{theor}(t_2) - h(t_2) < 0$. We assume, that the value Γ is an arc of visibility γ, $\gamma = \Gamma(t_1)$, under the following condition:

$$t_2 - t_1 \leq 4^{min} \qquad (9)$$

Let's explain the meaning of this condition. Due to the condition (9), the star appears in the twilight for a short time and then disappears. In the case of a smaller deletion of the star from the Sun x, twilight appearance of the star does not come, so we cannot determine the times t_1 and t_2. It means the conditions of visibility are not yet due. As an example, we calculated the arc of visibility of Venus and Mars, Fig. 8 and Fig. 9. On the horizontal axis we postponed the altitude of the planets $H(t)$, and the vertical axis value $h_{theor}(t) - h(t)$, which is a formal criterion of stellar visibility.

Calculations show that the value of the arc of visibility can vary subject to the brightness of the planet and the extinction coefficient. We are interested in getting the most typical values of the arcs of visibility, so we have taken the most typical of the planets magnitude just before conjunction with the Sun and used the extinction coefficient, which corresponds to a fairly clean air $k = 0.20$.

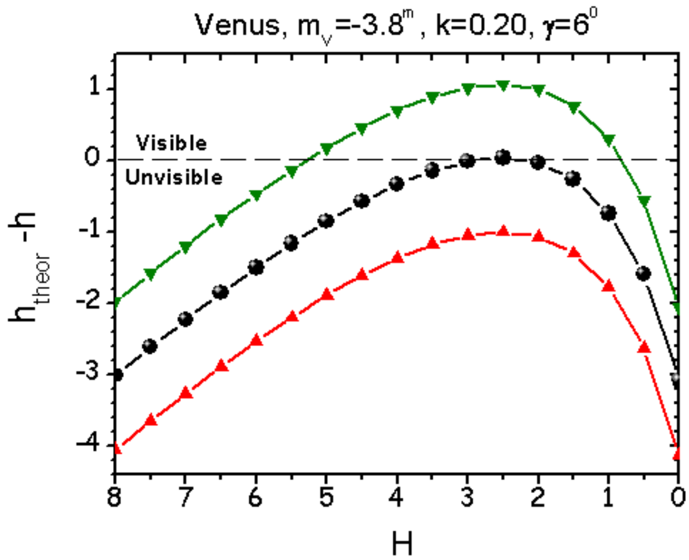

Fig. 8. *The calculation of the arc of visibility of Venus. On the horizontal axis the altitude H of the planet is postponed. The vertical axis corresponds to the difference between the theoretical value h_{theor} of solar depression below the horizon at which the planet can be seen and the actual solar depression h.*
Triangles facing down, shows the case when a planet is visible for a long time during twilight. Triangles facing up, marked the case when the evening visibility has passed some time ago. The circles marked case of an extremely visibility of the planet, for which we determine the arc of visibility.

Facing down triangles in both figures marked the case when the planet is visible in the twilight for quite a long time. It means that the planet can be seen next evening at the same state of atmospheric condition. The duration of visibility of the planet will decrease every day. Obviously, the case of an extremely visibility of the planet does not realize, so it does not makes no sense to define the arc of visibility. Facing up triangles denote the case when $h_{theor}(t)-h(t) < 0$. It means that during the night visibility of the planet had already passed and it disappeared behind the Sun. The heavy circles marked the case of an extremely visibility of the planet, for which we determine the arc of visibility.

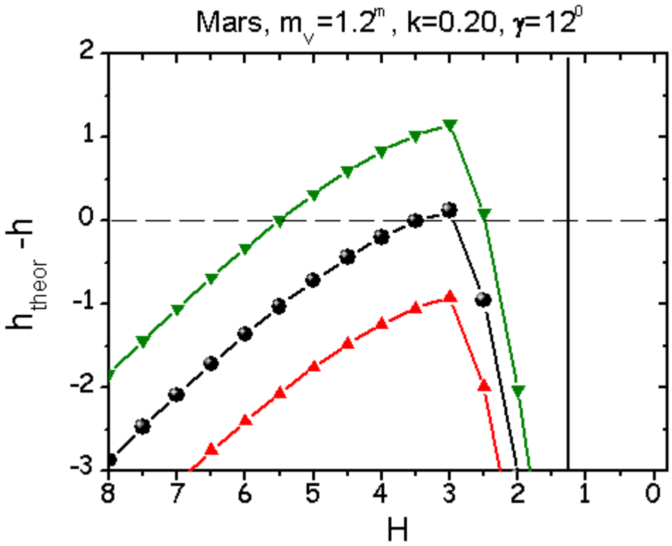

Fig. 9. *The calculation of the arc of visibility of Mars. Designations are similar.*

In Fig. 8 and Fig. 9 a difference profiles of the functions $h_{theor}(t) - h(t)$ of Venus and Mars is explained due to the fact that Venus remains brighter 4^m at the same set of the data parameters of the problem. In this case, to calculate the visibility conditions we use the equation $(4b)$. Mars has less magnitude; therefore its profile is calculated by the equations $(4b)$ and $(3b)$.

We compared the calculated values of the arcs of visibility of the bright planets with the values of the arcs of the studies of Ptolemy and Schoch. Ptolemy defined the arc of visibility on the basis of his own observations, which he tried to carry out under optimal weather conditions. About his own observations, Ptolemy told that summer observations in the vicinity of Cancer are more preferable, because at this time the air is clean and clear, and the slope of the zodiac is symmetric [in the east and west].

However, in the Almagest Ptolemy does not result observations themselves, therefore it is unknown how many observations for each planet he used. Schoch data are based on the Babylonian observations of the planets, which were held during different days of a year under different atmospheric conditions. Note that the average extinction coefficient in Egypt (where Ptolemy observed) is somewhat greater than in Mesopotamia. Therefore, the Ptolemy's arc of visibility must be greater than the Babylonian values. In the model calculation we used a value of the extinction coefficient equal to $k = 0.20$, which corresponds to a transparent atmosphere. We accepted the maximum value of extra-atmospheric brightness of the planets for the elongation at 10 degrees: Venus $m_V = -3.8^m$, Jupiter $m_V = -2.0^m$, Mercury $m_V = -1.5^m$, Saturn $m_V = 0.7^m$ and Mars $m_V = 1.2^m$. The comparison results are shown in Table 1.

Planet	**Ptolmey**	**Schoch**	**Model**
Venus	5	$5.8 \div 5.8\ u$ $5.2 \div 5.7\ b$ (15)	6
Jupiter	10	$7.4 \div 9.3$ (12)	8
Mercury	10	$11.1 \div 13.2\ u$ $9.5 \div 10.5\ b$ (30)	9
Saturn	11	$10.5 \div 13$ (9)	11
Mars	11.5	$14.2 \div 15.5$ (4)	12

Table 1. Comparison of the calculated values of the arcs of visibility with data of Ptolemy and Schoch. For Schoch's data, the total number of observations is given in curves. For Mercury and Venus the arcs at the moments of upper and lower conjunction of the planet are presented.

The simulation shows that the calculated arcs of visibility are in a good agreement with the results of Ptolemy and Schoch in case if the error of half a degree is ascribed for all quantities. Probably Ptolemy got too high value for the arc of visibility for Jupiter. Value of the arc of visibility of Mars was overrated due to the fact that Schoch used only four observations of this planet. We can assume that in all cases the state of the atmosphere did not allow to observe Mars at a lower value of the arc of visibility.

Besides, in this paper Schoch provides theoretical values of the arc of visibility for the heliacal rising (the first morning visibility - mfirst) and heliacal settings (the last night visibility - elast) of a stars. In Fig. 10 the results of comparison of our calculations and Schoch's data are given. Our calculations are corresponding to the value of extinction coefficient $k = 0.20$.

According to Schoch, the arc of the mfirst γ_R exceeds the arc of elast γ_S by 1^0. The difference between the arcs of visibility can be explained by the fact that the planet's position is known to the observer from the previous evenings for heliacal setting. For heliacal rising the place and date of the appearance of the star are unknown, which complicates the detection of a star. On the other hand, during the morning observations the atmosphere is more transparent, which allows to detect a star in more difficult conditions. Both of these factors holding in opposite directions compensate each other. The most likely explanation for the differences of the arcs of visibility is that the first morning visibility occurs when an observer records the detection of a star. Last evening visibility approaches at the first day when observer cannot detect a star during twilight. It means that on the date of observation the star was not found, and it last visibility occurred on the previous day. The difference of 1 day provides the difference of the arcs on 1^0 degree.

A theoretical calculation at value of the extinction coefficient $k = 0.20$ exactly coincides with Schoch's results for heliacal risings of the stars. However, our model allows us to calculate the arc of visibility over a wider range of magnitudes.

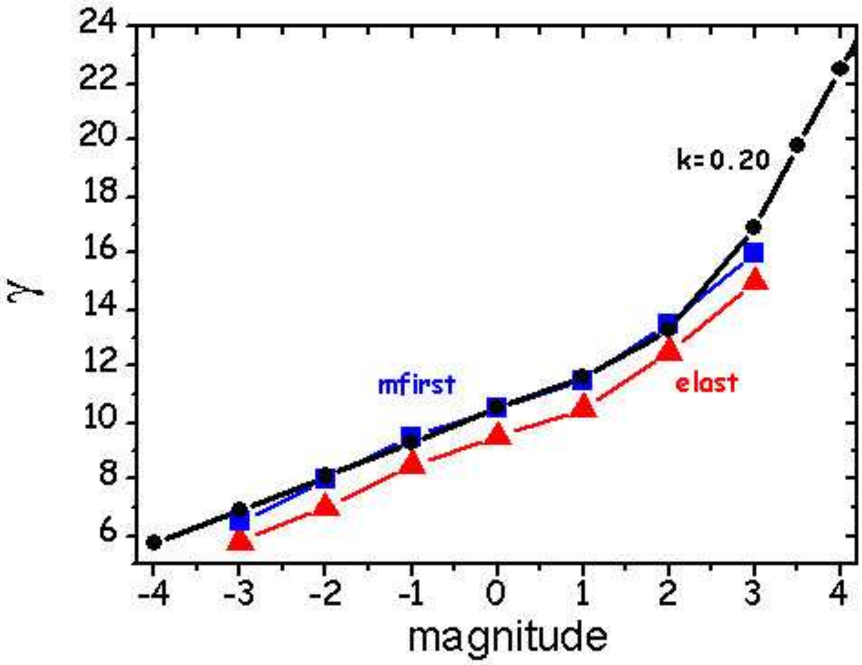

Fig. 10. *The dependence of the arc of visibility of magnitude $\gamma(m)$. Squares and triangles denote the theoretical estimates of Schoch for the first morning (mfirst) and the last evening (elast) visibility. Circles denotes theoretical calculation using our model.*

Next, let's check the model with the observational data of Schafer [1987]. To do this we used his observations from Table 1, in which the stars were registered at the limit of visibility. (In the table corresponding observations are marked with " $\geq$ "). We have calculated the arc of visibility on the date of observation for Schaefer's data, and compared it with the theoretical values, which were defined by the model. The calculation results are presented in Table 2.

Table 2 shows that in 7 out of 9 cases, theoretical values of the arcs of visibility γ_{theor} differ by no more than 2^0 from the values, which follow from the observations γ_{obs}. In addition, the obvious equality is true $\gamma_{theor} \leq \gamma_{obs}$. It means that it's really more difficult to discover a star in the sky than to predict the moment of it's appearance theoretically. In cases of No.4 and No.9 the difference between theoretical and experimental values is quite great, but the previous analysis shows that the model does not give such large errors. The most likely is that some observations are inaccurate. For example, estimates "the star is visible" (the ">") and "star is visible with difficulty" (the symbol "$\geq$") are subjective and depend on the observer. Therefore, in some cases errors are possible.

N	*Star*	*Event*	m_V	φ	k	*Date*	x	γ_{obs}	γ_{theor}
1	θ Gem	Rise	3.60	-30	0.25	20.08	47	22	21.5
2	α Leo	Set	1.33	19.8	0.13	08.08	14	11	10.5
3	κ Sco	Set	2.41	32	0.13	10.11	41	13.5	12
4	ρ Sgr	Rise	3.93	32	0.15	13.02	35	25	20
5	ρ Sgr	Rise	3.93	32	0.15	14.02	36	24	22
6	β Cap	Rise	3.08	32	0.18	15.02	22	18.5	17
7	β Cap	Rise	3.08	32	0.18	16.02	23	17.5	15.5
8	β Cap	Rise	3.08	32	0.18	17.02	24	17.5	15.5
9	ι Psc	Set	4.13	32	0.14	16.02	30	30	21

Table 2. Comparison of stellar arcs of visibility on the date of observation due to Schaefer's data with theoretical values which were defined by the model. Table m_V - extraatmospheric brightness of the star, φ - latitude of observation, k - extinction coefficient, x - distance between the star and the Sun.

This hypothesis is demonstrated by the fact that according to observation No.4 Schaefer's model gives the difference $\Delta_{pred} - \Delta_{obs} = -9^d$, and according to observation No.9 it provides $\Delta_{pred} - \Delta_{obs} = 14^d$. Let's note that according to Schaefer's results the maximum error $\Delta_{pred} - \Delta_{obs} = 22^d$ is achieved in observation No.3, where we got a good correspondence of arcs of visibility. In the rest 6 cases, Schafer's model fits well to his observation data because $\mid \Delta_{pred} - \Delta_{obs} \mid \leq 3^d$. Thus, in most cases, our model describes the observations of Schaefer very well.

5 Taking into account stellar spectra.

In all previous calculations, we assumed that the atmosphere is equally reduces the shine of stars, regardless of their spectrum. However, Forbes effect breaks this pattern. Forbes effect is that due to the bandwidth of the reaction and dependence on the wavelength of absorption, an attenuation coefficient depends on air mass. In other words, atmospheric absorption decreases subject to the decrease of the altitude (or the increase in atmospheric mass).

To estimate the magnitude of this effect we used the models of light attenuation of the Earth's atmosphere from A.I. Zakharov. The models were constructed as follows. Atmosphere (from 0 m to 100 km) was divided into layers of varying thickness. He used 10 layers, 100 m thick, 8 - 125 m, 5 - 200 m, 8 - 250 m, 6 - to 333 m, 6 - 500 m, 30 - 1 mile, 10 - 2 km, 6 - at 3.333 miles, 4 - to 5 km. For each layer from three models pressure, temperature, partial pressure of gases (H_2O, O_2, O_3) and the average concentration of aerosol was calculated. The models take into account the latitude and longitude of point location and dependence of distributions of all parameters on a season. Further, for the ray that connects the observer with a star, he calculated the optical depths for Rayleigh scattering, for aerosol scattering and for absorption of gases for each wavelength in steps of $50\mathring{A}$ in the range of 3025 to 9975Å. To obtain total atmospheric attenuation, light absorption and attenuation of light due to scattering were summed up in all layers. At the boundary of layers the refraction of light depending on the refractive index of air temperature, pressure and water vapor was calculated. Next, integration was made for the curve of the reaction of the eye and specific energy distribution in the spectrum of stars.

According to the observed dependence of atmospheric attenuation on the air mass a polynom was build. As a result, the author obtained three models for the atmosphere of Alexandria (Egypt), where a total absorption Δm is a function of atmospheric mass F_z and color index $B - V$.

1. Model of "clean" atmosphere. (Rayleigh atmosphere).

$$\Delta m = 0.1707F_Z(1 - 0.0655(B - V) + 0.00050F_Z(B - V) - 0.00040F_Z^2 + 0.00000720F_Z^3)$$

2. Model of "winter" aerosol.

$$\Delta m = 0.2616F_Z(1 - 0.0454(B - V) + 0.00030F_Z(B - V) - 0.00020F_Z^2 + 0.00001236F_Z^3)$$

3. Model of "average seasonal" aerosol.

$$\Delta m = 0.3397F_Z(1 - 0.0360(B - V) + 0.00008F_Z(B - V) - 0.00004F_Z^2 + 0.00001098F_Z^3)$$

In the model of "winter" aerosol the average concentration of aerosols during the three winter months was used. In the model of "average seasonal" ("average") aerosol average concentrations of aerosols in the atmosphere during season was used. Numerical factor in front of multiplier F_Z is the extinction coefficient corresponding to one air mass $k = \Delta m(F_Z = 1)$. Since the realization of the aerosol in the atmosphere can change, the extinction coefficients k_{Win} and k_{Ave} also vary around the average value. For the model of "clean" atmosphere it was assumed that the aerosol is completely absent and the scattering occurs at Rayleigh law. Therefore, the value of k_R is the minimum possible extinction coefficient, which is never realized in practice. The minimum value of the extinction coefficient for the middle latitudes is $k = 0.22$ [Kulikovskiy, 2009]. Smaller values are possible in the mountains, where the most dense atmosphere layers are excluded from integration.

The product $k \cdot F_Z$ determines the atmospheric absorption Δm for a white star of spectral class $A0$ similar to formula (6). An amendment of a stellar color is provided by the second and third terms of expression in the brackets which contain the factor $B - V$. The comparison of the models shows that when coefficient extinction increases an influence of a spectrum on the value of absorption decreases noticeably. For a white star spectral correction is equal to zero for any model, since the color index $\mid B - V \mid \approx 0$. In contrast, for red supergiants of spectral type M1.5Iab (Antares, Betelgeuse) $B-V=1.85^m$ we get a maximum value of the color correction. The last two terms, which contain factors F_Z^2 and F_Z^3, show the most significant effect on the value of attenuation for a star in the vicinity of the horizon. However, in most cases can be neglected. Let's consider the influence of the effect of color correction terms using concrete examples.

Example No.5. *Influence of the spectral corrections on the heliacal visibility of the planets.*

According to our calculations, Venus at $m_V=-3.8^m$ and $k=0.20$ has an arc of visibility $\gamma=6.1^0$ at the altitude $H=2.5^0$. The color index of Venus slightly depends on the phase angle and reaches a minimum value $B-V\approx 0.7^m$ in the upper and lower conjunctions [Irvine, 1968]. Then, taking in account spectral characteristics, its magnitude will exceed a theoretical value on $dm\approx 0.27^m$, which will reduce the arc of visibility up to $\gamma'=5.7^0$. Shortly before conjunction with the Sun, Jupiter has a magnitude $m_V=-2.0^m$, and its arc of visibility without correction is $\gamma=8.3^0$. Assuming that color index of the planet is $B-V\approx 0.85^m$ [Irvine, 1968] we get the value of the arc of visibility about $\gamma=7.8^0$. For Mars, at $m_V=1.2^m$ and $k=0.20$ the arc of visibility is $\gamma=12.1^0$. Assuming that the color index of Mars is $B-V\approx 1.3^m$ we obtain, the corrected value of the arc of visibility equal to $\gamma=11.4^0$. For Antares, at $m_V=0.96^m$, $k=0.20$ and $B-V=1.85^m$ we have the greatest difference $\gamma=11.8^0$ and $\gamma'=11.0^0$. Finally, for a relatively faint red star at $m_V=3.5^m$, $k=0.20$ and $B-V=1.85^m$ we get the difference $\gamma=14.8^0$ and $\gamma=14.4^0$.

It follows from these estimates that for bright yellow stars with a value of color index of about $B-V=0.60\div 1.0^m$ the consideration of atmospheric effects leads to a decrease the arc of visibility at 0.5 degree. Thus, we can calculate the arc of visibility of stars using a simple model (6) and then adjust it with the amendments. More accurate calculations have no meaning by virtue of errors in input data and the model itself. For the brightest red stars, which has color index $B-V>1.0^m$, a value of correction is about $0.5\div 0.7$ degree. With a decreas of magnitude the value of a correction also decreases. For the objects with stellar magnitude of $\sim 4^m$ a correction can be neglected even for red stars. This is due to the fact that the faint objects begin to be noticeable quite at high altitudes when the Forbes effect is small. Finally, let's note that in all these examples, we used the model, which is most strongly dependent on the color index of stars. In practice, the presence of aerosols reduces the dependence of value of atmospheric absorption on the spectral characteristics.

Conclusion. Theoretical estimates of the arc of visibility of the planets and stars have shown a good agreement with the data of Ptolemy, Schoch and with observations of Schaefer. This implies that the extrapolation of the equation (7) is true for $x<20^0$ and our model successfully describes the twilight visibility of stars. This model can be used to study the twilight visibility of stars in the field of ancient astronomy.

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