The main part of the perturbation function in the restricted three-body problem

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Abstract. This paper gives the main part of the perturbation function in the restricted three-body problem up to fourth order, concerning the eccentricities and the inclination. This result is obtained by the approach that is described in our paper "On the Movement of the Asteroid 108 Hecuba". The coefficients are represented by the generalized binomial transform and the Laplace's coefficients of the ratio of the Hecuba's and the Jupiter's semi-major orbital axes.
Key words: perturbation function, generalized binomial transform, Laplace's coefficients

Главната част от пертурбационната функция в ограничената задача за три тела

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Тази статия дава главната част от пертурбационната функция в ограничената задача за три тела до четвърти порядък, обхващащ екцентрицитетите и инклинацията. Този резултат е получен чрез подход, който е описан в нашата работа "'Върху движението на астероида 108 Хекуба"'. Коефициентите са представени чрез обобщено биномиално преобразуване на Лапласовите коефициенти на отношението на големите полуоси на орбитите на Хекуба и Юпитер.

Introduction

To determine the motion of the asteroid Hecuba [Popoff K.,1912] we derive the perturbation function following [Smart W.,1953]. This paper is the continuation of our paper "On the Movement of the Asteroid 108 Hecuba" [Borisov B., Shkodrov V.,2006]. There we show the application that computes the main part of the perturbation function in the restricted three-body problem up to the eighth order concerning the Laplace's coefficients [Smart W.,1953] and up to the fourth order concerning the eccentricities and the inclination. Using the generalized binomial transform [Borisov B., Shkodrov V., 2007] we express here the main part of the perturbation function, that includes all terms concerning the Laplace's coefficients. Other expression of perturbation function is represented in [Leverrier U.,1855,1856], [Tisserand F.,1889].

1 Symbols

R – the main part of the perturbation function in the restricted three-body problem up to fourth order concerning the eccentricities and the inclination.
R_1 is a determined by the following relation:

$$R = \frac{\gamma m_1}{a_1} R_1 \tag{1}$$

where γ is the gravitational constant, m_1 and a_1 are the mass and the semi-major axis of Jupiter.
e – the eccentricity of Hecuba.
e_1 – the eccentricity of Jupiter.
j is a quantity that depends on the inclination i between two orbits by the relation:

$$j = 2 \sin i/2 \tag{2}$$

M – the mean anomaly of Hecuba.
M_1 – the mean anomaly of Jupiter.
φ – the difference between the mean longitudes of Hecuba and Jupiter.
ω – the argument of pericenter of Hecuba.
$a_n^k\left(d_0, d_1, d_2, ..., d_r\right)$ is determined by the following relation:

$$a_n^k\left(d_0, d_1, d_2, ..., d_r\right) = b_n\left(d_0, d_1, d_2, ..., d_r\right)\alpha^k \frac{d^k B_{|n|}^{1/2}}{d\alpha^k}, \qquad \begin{matrix} n = 0, \pm 1, \pm 2, \pm 3, ... \\ k = 0, 1, 2, 3, 4 \end{matrix} \tag{3}$$

where $B_n^{1/2}\left(\alpha\right)$ is the Laplace's coefficients [Smart W.,1953] of the ratio of the Hecuba's and the Jupiter's semi-major axes α and $b_n\left(d_0, d_1, d_2, ..., d_r\right)$ is the n^{th} term in the generalized binomial transform [Borisov B., Shkodrov V., 2007] of r-tuple $d_0, d_1, d_2, ..., d_r$ [Weisstein E.,1995]:

$$b_n\left(d_0, d_1, d_2, ..., d_r\right) = \sum_{m=0}^{r} \binom{n}{m} d_m, \qquad n = 0, \pm 1, \pm 2, \pm 3, ... \tag{4}$$

For example:

$$a_{-2}^1\left(-\frac{1}{8}, \frac{5}{16}, \frac{5}{8}\right) = b_{-2}\left(-\frac{1}{8}, \frac{5}{16}, \frac{5}{8}\right)\alpha\frac{dB_2^{1/2}}{d\alpha} = \frac{9}{8}\alpha\frac{dB_2^{1/2}}{d\alpha}$$

2 Result

$$\begin{aligned}
R_1 = \sum_{n=-\infty}^{\infty} & \left\{a_n^0\left(\frac{1}{2}\right) - \left(e^2 + e_1^2\right)\left[a_n^0\left(0, \frac{1}{2}, 1\right) - a_n^1\left(\frac{1}{4}\right) - a_n^2\left(\frac{1}{8}\right)\right] + \right. \\
& + j^2\left[a_n^0\left(0, \frac{1}{8}, \frac{1}{4}\right) - a_n^1\left(\frac{1}{4}\right) - a_n^2\left(\frac{1}{8}\right)\right] - \left(e^2 + e_1^2\right) j^2\left[a_n^0\left(0, \frac{1}{8}, \frac{7}{4}, \frac{9}{2}, 3\right) - \right. \\
& - a_n^1\left(-\frac{1}{8}, \frac{5}{16}, \frac{5}{8}\right) - a_n^2\left(-\frac{7}{16}, \frac{5}{32}, \frac{5}{16}\right) + a_n^3\left(\frac{1}{4}\right) + a_n^4\left(\frac{1}{32}\right)\Big] + \\
& + e^2 e_1^2\left[a_n^0\left(0, \frac{1}{2}, 7, 18, 12\right) - a_n^1\left(-\frac{1}{8}, \frac{1}{2}, 1\right) - a_n^2\left(-\frac{7}{16}, \frac{1}{4}, \frac{1}{2}\right) + a_n^3\left(\frac{1}{4}\right) + \right. \\
& + a_n^4\left(\frac{1}{32}\right)\Big] + e^4\left[a_n^0\left(0, \frac{7}{128}, \frac{103}{64}, \frac{9}{2}, 3\right) - a_n^1\left(0, \frac{1}{16}, \frac{1}{8}\right) - a_n^2\left(0, \frac{1}{16}, \frac{1}{8}\right) + \right. \\
& + a_n^3\left(\frac{1}{32}\right) + a_n^4\left(\frac{1}{128}\right)\Big] + e_1^4\left[a_n^0\left(0, -\frac{1}{128}, \frac{95}{64}, \frac{9}{2}, 3\right) - a_n^1\left(-\frac{3}{16}, \frac{3}{16}, \frac{3}{8}\right) - \right. \\
& - a_n^2\left(-\frac{9}{32}, \frac{1}{16}, \frac{1}{8}\right) + a_n^3\left(\frac{3}{32}\right) + a_n^4\left(\frac{1}{128}\right)\Big] + j^4\left[a_n^0\left(0, \frac{1}{32}, \frac{5}{32}, \frac{9}{32}, \frac{3}{16}\right) - \right. \\
& - a_n^1\left(0, \frac{1}{32}, \frac{1}{16}\right) - a_n^2\left(-\frac{3}{32}, \frac{1}{64}, \frac{1}{32}\right) + a_n^3\left(\frac{1}{16}\right) + a_n^4\left(\frac{1}{128}\right)\Big]\Big\}\cos\left(n\varphi\right) + \\
& + e\left\{a_n^0\left(0, 1\right) - a_n^1\left(\frac{1}{2}\right) - e^2\left[a_n^0\left(0, \frac{5}{4}, \frac{17}{4}, 3\right) - a_n^1\left(\frac{3}{16}, \frac{11}{16}, \frac{1}{2}\right) + a_n^2\left(\frac{1}{8}, -\frac{1}{8}\right) + \right. \right. \\
& + a_n^3\left(\frac{1}{16}\right)\Big] - e_1^2\left[a_n^0\left(0, 1, 6, 6\right) - a_n^1\left(-\frac{1}{4}, 1, 1\right) + a_n^2\left(\frac{1}{2}, -\frac{1}{4}\right) + a_n^3\left(\frac{1}{8}\right)\Big] +
\end{aligned}$$

$$+ j^2\left[a_n^0\left(0,\frac{1}{4},\frac{3}{2},\frac{3}{2}\right)-a_n^1\left(-\frac{1}{4},\frac{5}{8},\frac{1}{4}\right)+a_n^2\left(\frac{1}{2},-\frac{1}{4}\right)+a_n^3\left(\frac{1}{8}\right)\right]\Bigg\}\cos\left(M+n\varphi\right)+$$
$$+ e^2\left\{a_n^0\left(0,\frac{9}{8},1\right)-a_n^1\left(\frac{1}{4},\frac{1}{2}\right)+a_n^2\left(\frac{1}{8}\right)-e^2\left[a_n^0\left(0,\frac{27}{16},\frac{89}{12},\frac{39}{4},4\right)-\right.\right.$$
$$- a_n^1\left(\frac{1}{6},\frac{55}{48},2,1\right)+a_n^2\left(\frac{1}{8},\frac{3}{32}\right)+a_n^3\left(0,\frac{1}{24}\right)-a_n^4\left(\frac{1}{96}\right)\Bigg]-$$
$$- e_1^2\left[a_n^0\left(0,\frac{9}{8},\frac{43}{4},\frac{87}{4},12\right)-a_n^1\left(-\frac{1}{8},\frac{17}{16},4,3\right)+a_n^2\left(\frac{1}{16},\frac{11}{32}\right)-a_n^3\left(\frac{1}{8},-\frac{1}{8}\right)-\right.$$
$$- a_n^4\left(\frac{1}{32}\right)\Bigg]+j^2\left[a_n^0\left(0,\frac{9}{32},\frac{43}{16},\frac{87}{16},3\right)-a_n^1\left(-\frac{1}{8},\frac{1}{2},\frac{11}{8},\frac{3}{4}\right)+\right.$$
$$+ a_n^2\left(\frac{1}{16},\frac{1}{4},-\frac{3}{16}\right)+a_n^3\left(-\frac{1}{8},\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\Bigg]\Bigg\}\cos\left(2M+n\varphi\right)+$$
$$+ e^3\left[a_n^0\left(0,\frac{4}{3},\frac{9}{4},1\right)-a_n^1\left(\frac{3}{16},\frac{13}{16},\frac{1}{2}\right)+a_n^2\left(\frac{1}{8},\frac{1}{8}\right)-a_n^3\left(\frac{1}{48}\right)\right]\cos\left(3M+n\varphi\right)+$$
$$+ e^4\left[a_n^0\left(0,\frac{625}{384},\frac{755}{192},\frac{27}{8},1\right)-a_n^1\left(\frac{1}{6},\frac{109}{96},\frac{11}{8},\frac{1}{2}\right)+a_n^2\left(\frac{1}{8},\frac{17}{64},\frac{1}{8}\right)-\right.$$
$$- a_n^3\left(\frac{1}{32},\frac{1}{48}\right)+a_n^4\left(\frac{1}{384}\right)\Bigg]\cos\left(4M+n\varphi\right)+$$
$$+ e_1\left\{a_n^0\left(\frac{1}{2},-1\right)+a_n^1\left(\frac{1}{2}\right)+e_1^2\left[a_n^0\left(-\frac{1}{16},-\frac{1}{16},\frac{5}{4},3\right)-\right.\right.$$
$$- a_n^1\left(-\frac{7}{16},\frac{5}{16},\frac{1}{2}\right)+a_n^2\left(\frac{7}{16},-\frac{1}{2}\right)+a_n^3\left(\frac{1}{16}\right)\Bigg]+e^2\left[a_n^0\left(0,\frac{1}{2},5,6\right)-\right.$$
$$- a_n^1\left(-\frac{1}{2},1,1\right)+a_n^2\left(\frac{5}{8},-\frac{1}{4}\right)+a_n^3\left(\frac{1}{8}\right)\Bigg]-j^2\left[a_n^0\left(0,\frac{1}{8},\frac{5}{4},\frac{3}{2}\right)-\right.$$
$$- a_n^1\left(-\frac{1}{2},\frac{5}{8},\frac{1}{4}\right)+a_n^2\left(\frac{5}{8},-\frac{1}{4}\right)+a_n^3\left(\frac{1}{8}\right)\Bigg]\Bigg\}\cos\left(M_1+n\varphi\right)+$$
$$+ e_1^2\left\{a_n^0\left(\frac{1}{2},-\frac{5}{8},1\right)+a_n^1\left(\frac{3}{4},-\frac{1}{2}\right)+a_n^2\left(\frac{1}{8}\right)-e_1^2\left[a_n^0\left(\frac{1}{6},-\frac{5}{48},-\frac{1}{12},\frac{5}{4},4\right)-\right.\right.$$
$$- a_n^1\left(\frac{1}{3},-\frac{11}{48},0,1\right)-a_n^2\left(\frac{5}{8},-\frac{9}{32}\right)-a_n^3\left(\frac{1}{6},-\frac{1}{24}\right)-a_n^4\left(\frac{1}{96}\right)\Bigg]-$$
$$- e^2\left[a_n^0\left(0,-\frac{1}{8},\frac{5}{4},\frac{45}{4},12\right)-a_n^1\left(\frac{5}{8},-\frac{13}{16},2,3\right)-a_n^2\left(\frac{17}{16},-\frac{25}{32}\right)-\right.$$
$$- a_n^3\left(\frac{3}{8},-\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\Bigg]+j^2\left[a_n^0\left(0,-\frac{1}{32},\frac{5}{16},\frac{45}{16},3\right)-a_n^1\left(\frac{5}{8},-\frac{5}{8},\frac{7}{8},\frac{3}{4}\right)-\right.$$
$$- a_n^2\left(\frac{17}{16},-\frac{11}{16},\frac{3}{16}\right)+a_n^3\left(-\frac{3}{8},\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\Bigg]\Bigg\}\cos\left(2M_1+n\varphi\right)+$$
$$+ e_1^3\left[a_n^0\left(\frac{9}{16},-\frac{31}{48},\frac{3}{4},-1\right)+a_n^1\left(\frac{17}{16},-\frac{13}{16},\frac{1}{2}\right)+a_n^2\left(\frac{5}{16},-\frac{1}{8}\right)+\right.$$
$$+ a_n^3\left(\frac{1}{48}\right)\Bigg]\cos\left(3M_1+n\varphi\right)+$$
$$+ e_1^4\left[a_n^0\left(\frac{2}{3},-\frac{283}{384},\frac{155}{192},-\frac{7}{8},1\right)+a_n^1\left(\frac{71}{48},-\frac{115}{96},\frac{7}{8},-\frac{1}{2}\right)+\right.$$
$$+ a_n^2\left(\frac{19}{32},-\frac{21}{64},\frac{1}{8}\right)+a_n^3\left(\frac{7}{96},-\frac{1}{48}\right)+a_n^4\left(\frac{1}{384}\right)\Bigg]\cos\left(4M_1+n\varphi\right)+$$

$$
\begin{aligned}
&+ ee_1\left\{-a_n^0\left(0,\frac{1}{2},2\right)+a_n^1\left(-\frac{1}{2},1\right)-a_n^2\left(\frac{1}{4}\right)+\right.\\
&+e^2\left[a_n^0\left(0,\frac{5}{8},\frac{71}{8},\frac{81}{4},12\right)-a_n^1\left(-\frac{3}{16},\frac{13}{16},4,3\right)+a_n^2\left(-\frac{3}{32},\frac{17}{32}\right)+\right.\\
&+\left.a_n^3\left(-\frac{3}{16},\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\right]+e_1^2\left[a_n^0\left(0,-\frac{1}{8},\frac{19}{8},\frac{51}{4},12\right)-\right.\\
&-a_n^1\left(\frac{3}{16},-\frac{5}{16},2,3\right)+a_n^2\left(-\frac{21}{32},\frac{19}{32}\right)+a_n^3\left(-\frac{5}{16},\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\Big]-\\
&-j^2\left[a_n^0\left(0,\frac{1}{8},\frac{11}{4},\frac{33}{4},6\right)-a_n^1\left(\frac{1}{4},-\frac{1}{8},\frac{9}{4},\frac{3}{2}\right)-a_n^2\left(\frac{7}{8},-\frac{15}{16},\frac{3}{8}\right)-\right.\\
&-\left.\left.a_n^3\left(\frac{1}{2},-\frac{1}{4}\right)-a_n^4\left(\frac{1}{16}\right)\right]\right\}\cos\left(M+M_1+n\varphi\right)+\\
&+ee_1\left\{a_n^0\left(0,\frac{3}{2},2\right)-a_n^1\left(\frac{1}{2}\right)-a_n^2\left(\frac{1}{4}\right)-e^2\left[a_n^0\left(0,\frac{15}{8},\frac{105}{8},\frac{93}{4},12\right)-\right.\right.\\
&-a_n^1\left(\frac{3}{16},\frac{15}{16},\frac{3}{4}\right)-a_n^2\left(-\frac{3}{32},\frac{17}{32},\frac{1}{2}\right)+a_n^3\left(\frac{3}{16}\right)+a_n^4\left(\frac{1}{32}\right)\Big]-\\
&-e_1^2\left[a_n^0\left(0,\frac{7}{4},\frac{103}{8},\frac{93}{4},12\right)-a_n^1\left(-\frac{3}{16},\frac{19}{16},\frac{5}{4}\right)-a_n^2\left(-\frac{21}{32},\frac{17}{32},\frac{1}{2}\right)+\right.\\
&+a_n^3\left(\frac{5}{16}\right)+a_n^4\left(\frac{1}{32}\right)\Big]+j^2\left[a_n^0\left(0,\frac{3}{8},\frac{17}{4},\frac{39}{4},6\right)-a_n^1\left(-\frac{1}{4},\frac{7}{8},\frac{5}{4}\right)-\right.\\
&-\left.\left.a_n^2\left(-\frac{7}{8},\frac{7}{16},\frac{5}{8}\right)+a_n^3\left(\frac{1}{2}\right)+a_n^4\left(\frac{1}{16}\right)\right]\right\}\cos\left(M-M_1+n\varphi\right)+\\
&+ee_1^2\left[a_n^0\left(0,-\frac{1}{8},\frac{3}{4},3\right)-a_n^1\left(\frac{5}{8},-\frac{13}{16},\frac{3}{2}\right)+a_n^2\left(-\frac{1}{2},\frac{3}{8}\right)-\right.\\
&-\left.a_n^3\left(\frac{1}{16}\right)\right]\cos\left(M+2M_1+n\varphi\right)+\\
&+ee_1^2\left[a_n^0\left(0,\frac{17}{8},\frac{21}{4},3\right)+a_n^1\left(-\frac{5}{8},\frac{3}{16},\frac{1}{2}\right)-a_n^2\left(\frac{1}{2},\frac{1}{8}\right)-\right.\\
&-\left.a_n^3\left(\frac{1}{16}\right)\right]\cos\left(M-2M_1+n\varphi\right)-\\
&-e^2e_1\left[a_n^0\left(0,\frac{9}{16},\frac{15}{4},3\right)-a_n^1\left(-\frac{1}{4},\frac{13}{16},\frac{3}{2}\right)-a_n^2\left(\frac{1}{16},-\frac{3}{8}\right)-\right.\\
&-\left.a_n^3\left(\frac{1}{16}\right)\right]\cos\left(2M+M_1+n\varphi\right)+\\
&+e^2e_1\left[a_n^0\left(0,\frac{27}{16},\frac{19}{4},3\right)-a_n^1\left(\frac{1}{4},\frac{11}{16},\frac{1}{2}\right)+a_n^2\left(\frac{1}{16},-\frac{1}{8}\right)+\right.\\
&+\left.a_n^3\left(\frac{1}{16}\right)\right]\cos\left(2M-M_1+n\varphi\right)+\\
&+ee_1^3\left[a_n^0\left(0,\frac{1}{12},-\frac{5}{24},\frac{3}{4},4\right)-a_n^1\left(-\frac{13}{16},\frac{47}{48},-\frac{5}{4},2\right)+a_n^2\left(\frac{27}{32},-\frac{23}{32},\frac{1}{2}\right)-\right.\\
&-\left.a_n^3\left(-\frac{3}{16},\frac{1}{12}\right)+a_n^4\left(\frac{1}{96}\right)\right]\cos\left(M+3M_1+n\varphi\right)+\\
&+ee_1^3\left[a_n^0\left(0,\frac{71}{24},\frac{247}{24},\frac{45}{4},4\right)+a_n^1\left(-\frac{13}{16},\frac{25}{48},2,1\right)-a_n^2\left(\frac{27}{32},\frac{11}{32}\right)-\right.
\end{aligned}
$$

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\begin{aligned}
&- a_n^3\left(\frac{3}{16},\frac{1}{24}\right) - a_n^4\left(\frac{1}{96}\right)\Big]\cos\left(M-3M_1+n\varphi\right)+\\
&+ e^2e_1^2\left[a_n^0\left(0,-\frac{9}{4},\frac{35}{32},\frac{15}{2},6\right) - a_n^1\left(\frac{5}{16},-\frac{1}{2},\frac{3}{2},3\right) + a_n^2\left(\frac{1}{32},-\frac{3}{16},\frac{3}{4}\right)+\right.\\
&+ a_n^3\left(\frac{1}{8},-\frac{1}{8}\right) + a_n^4\left(\frac{1}{64}\right)\Big]\cos\left(2M+2M_1+n\varphi\right)+\\
&+ e^2e_1^2\left[a_n^0\left(0,\frac{153}{64},\frac{341}{32},\frac{57}{4},6\right) - a_n^1\left(\frac{5}{16},\frac{13}{16},\frac{1}{2}\right) - a_n^2\left(-\frac{1}{32},\frac{13}{32},\frac{1}{4}\right)+\right.\\
&+ a_n^3\left(\frac{1}{8}\right) + a_n^4\left(\frac{1}{64}\right)\Big]\cos\left(2M-2M_1+n\varphi\right)-\\
&- e^3e_1\left[a_n^0\left(0,\frac{2}{3},\frac{145}{24},\frac{37}{4},-4\right) - a_n^1\left(-\frac{3}{16},\frac{41}{48},\frac{13}{4},2\right) + a_n^2\left(-\frac{3}{32},\frac{15}{32},\frac{1}{2}\right)-\right.\\
&- a_n^3\left(\frac{1}{48},\frac{1}{12}\right) + a_n^4\left(\frac{1}{96}\right)\Big]\cos\left(3M+M_1+n\varphi\right)+\\
&+ e^3e_1\left[a_n^0\left(0,2,\frac{199}{24},\frac{41}{4},4\right) - a_n^1\left(\frac{3}{16},\frac{55}{48},2,1\right) + a_n^2\left(\frac{3}{32},\frac{1}{32}\right)+\right.\\
&+ a_n^3\left(\frac{1}{48},\frac{1}{24}\right) - a_n^4\left(\frac{1}{96}\right)\Big]\cos\left(3M-M_1+n\varphi\right)-\\
&- e^2j^2\left[a_n^0\left(0,\frac{21}{32},\frac{45}{16},\frac{117}{32},\frac{3}{2}\right) - a_n^1\left(\frac{3}{8},\frac{9}{32},-\frac{7}{16},-\frac{3}{8}\right) - a_n^2\left(\frac{9}{16},\frac{29}{64},\frac{3}{32}\right)-\right.\\
&- a_n^3\left(\frac{3}{16},\frac{1}{16}\right) - a_n^4\left(\frac{1}{64}\right)\Big]\cos\left(2\omega+n\varphi\right)+\\
&+ ej^2\left[a_n^0\left(0,\frac{3}{4},\frac{3}{2},\frac{3}{4}\right) - a_n^1\left(\frac{5}{8},\frac{1}{8},-\frac{1}{8}\right) - a_n^2\left(\frac{1}{2},\frac{1}{8}\right)-\right.\\
&- a_n^3\left(\frac{1}{16}\right)\Big]\cos\left(M+2\omega+n\varphi\right)-\\
&- j^2\left\{a_n^0\left(0,\frac{1}{4},\frac{1}{4}\right) - a_n^1\left(\frac{1}{4}\right) - a_n^2\left(\frac{1}{8}\right) - e^2\left[a_n^0\left(0,\frac{9}{4},\frac{15}{2},\frac{33}{4},3\right)-\right.\right.\\
&- a_n^1\left(\frac{7}{8},\frac{11}{8},\frac{5}{8}\right) - a_n^2\left(\frac{1}{16},\frac{11}{16},\frac{5}{16}\right) + a_n^3\left(\frac{1}{4}\right) + a_n^4\left(\frac{1}{32}\right)\Big] -\\
&- e_1^2\left[a_n^0\left(0,\frac{1}{4},\frac{5}{2},\frac{21}{4},3\right) - a_n^1\left(-\frac{1}{8},\frac{3}{8},\frac{5}{8}\right) - a_n^2\left(-\frac{7}{16},\frac{3}{16},\frac{5}{16}\right) + a_n^3\left(\frac{1}{4}\right)+\right.\\
&+ a_n^4\left(\frac{1}{32}\right)\Big] + j^2\left[a_n^0\left(0,\frac{1}{8},\frac{7}{16},\frac{9}{16},\frac{1}{4}\right) - a_n^1\left(\frac{1}{48},\frac{5}{48},\frac{1}{12}\right)-\right.\\
&\left.\left.- a_n^2\left(-\frac{11}{96},\frac{5}{96},\frac{1}{24}\right) + a_n^3\left(\frac{1}{12}\right) + a_n^4\left(\frac{1}{96}\right)\Big]\right\}\cos\left(2M+2\omega+n\varphi\right)-\\
&- ej^2\left[a_n^0\left(0,\frac{3}{4},\frac{3}{2},\frac{3}{4}\right) - a_n^1\left(\frac{3}{8},\frac{3}{8},\frac{1}{8}\right) - a_n^2\left(0,\frac{1}{8}\right)+\right.\\
&+ a_n^3\left(\frac{1}{16}\right)\Big]\cos\left(3M+2\omega+n\varphi\right)-\\
&- e^2j^2\left[a_n^0\left(0,\frac{51}{32},\frac{75}{16},\frac{147}{32},\frac{3}{2}\right) - a_n^1\left(\frac{1}{2},\frac{35}{32},\frac{17}{16},\frac{3}{8}\right) - a_n^2\left(-\frac{1}{8},\frac{7}{64},\frac{3}{32}\right)+\right.\\
&+ a_n^3\left(\frac{1}{16},\frac{1}{16}\right) - a_n^4\left(\frac{1}{64}\right)\Big]\cos\left(4M+2\omega+n\varphi\right)-
\end{aligned}
$$

$$
\begin{aligned}
&- ee_1j^2\left[a_n^0\left(0,\frac{3}{8},\frac{15}{4},\frac{51}{8},3\right)-a_n^1\left(-\frac{5}{8},1,\frac{7}{8}\right)-a_n^2\left(-\frac{17}{6},\frac{3}{8},\frac{5}{16}\right)+\right.\\
&\left.+\ a_n^3\left(\frac{3}{8}\right)+a_n^4\left(\frac{1}{32}\right)\right]\cos\left(M+M_1+2\omega+n\varphi\right)+\\
&+ ee_1j^2\left[a_n^0\left(0,\frac{9}{8},\frac{21}{4},\frac{57}{8},3\right)+a_n^1\left(-\frac{5}{8},-\frac{1}{2},\frac{7}{8},\frac{3}{4}\right)-a_n^2\left(\frac{17}{16},\frac{7}{8},\frac{3}{16}\right)-\right.\\
&\left.-\ a_n^3\left(\frac{3}{8},\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\right]\cos\left(M-M_1+2\omega+n\varphi\right)+\\
&+ e_1j^2\left[a_n^0\left(0,\frac{1}{8},\frac{7}{8},\frac{3}{4}\right)-a_n^1\left(-\frac{1}{4},\frac{3}{8},\frac{1}{8}\right)+a_n^2\left(\frac{5}{16},-\frac{1}{8}\right)+\right.\\
&\left.+\ a_n^3\left(\frac{1}{6}\right)\right]\cos\left(2M+M_1+2\omega+n\varphi\right)-\\
&- e_1j^2\left[a_n^0\left(0,\frac{3}{8},\frac{9}{8},\frac{3}{4}\right)-a_n^1\left(\frac{1}{4},\frac{1}{8},-\frac{1}{8}\right)-a_n^2\left(\frac{5}{16},\frac{1}{8}\right)-\right.\\
&\left.-\ a_n^3\left(\frac{1}{16}\right)\right]\cos\left(2M-M_1+2\omega+n\varphi\right)-\\
&- e_1^2j^2\left[a_n^0\left(0,-\frac{1}{32},\frac{1}{4},\frac{57}{32},\frac{3}{2}\right)-a_n^1\left(\frac{5}{16},-\frac{11}{32},\frac{9}{16},\frac{3}{8}\right)-a_n^2\left(\frac{17}{32},-\frac{23}{64},\frac{3}{32}\right)-\right.\\
&\left.-\ a_n^3\left(\frac{3}{16},-\frac{1}{16}\right)-a_n^4\left(\frac{1}{64}\right)\right]\cos\left(2M+2M_1+2\omega+n\varphi\right)-\\
&- e_1^2j^2\left[a_n^0\left(0,\frac{17}{32},\frac{5}{2},\frac{111}{32},\frac{3}{2}\right)-a_n^1\left(\frac{5}{16},\frac{7}{32},-\frac{7}{16},-\frac{3}{8}\right)-a_n^2\left(\frac{17}{32},\frac{27}{64},\frac{3}{32}\right)-\right.\\
&\left.-\ a_n^3\left(\frac{3}{16},\frac{1}{16}\right)-a_n^4\left(\frac{1}{64}\right)\right]\cos\left(2M-2M_1+2\omega+n\varphi\right)+\\
&+ ee_1j^2\left[a_n^0\left(0,\frac{3}{8},\frac{15}{4},\frac{51}{8},3\right)-a_n^1\left(-\frac{3}{8},\frac{3}{4},\frac{13}{8},\frac{3}{4}\right)+a_n^2\left(\frac{3}{16},\frac{1}{4},-\frac{3}{16}\right)+\right.\\
&\left.+\ a_n^3\left(-\frac{1}{8},\frac{1}{8}\right)-a_n^4\left(\frac{1}{32}\right)\right]\cos\left(3M+M_1+2\omega+n\varphi\right)-\\
&- ee_1j^2\left[a_n^0\left(0,\frac{9}{8},\frac{21}{4},\frac{57}{8},3\right)-a_n^1\left(\frac{3}{8},\frac{3}{4},\frac{3}{8}\right)-a_n^2\left(\frac{3}{16},\frac{1}{2},\frac{5}{16}\right)+a_n^3\left(\frac{1}{8}\right)+\right.\\
&\left.+\ a_n^4\left(\frac{1}{32}\right)\right]\cos\left(3M-M_1+2\omega+n\varphi\right)+\\
&+ j^4\left[a_n^0\left(0,\frac{1}{16},\frac{3}{16},\frac{3}{16},\frac{1}{16}\right)-a_n^1\left(\frac{1}{48},\frac{1}{24},\frac{1}{48}\right)-\right.\\
&\left.-\ a_n^2\left(-\frac{1}{48},\frac{1}{48},\frac{1}{96}\right)+a_n^3\left(\frac{1}{48}\right)+a_n^4\left(\frac{1}{384}\right)\right]\cos\left(4M+4\omega+n\varphi\right)
\end{aligned}
\tag{5}
$$

Conclusion

This result is the generalization concerning the difference between the mean longitudes. The generalization concerning the mean anomalies can be derived if the series contain more terms. Also it is interesting to obtain the relations between the r-tuples in the result.

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