Correlations of parameters of Titius-Bode law with parameters of exoplanet and satellite systems

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Abstract. The Titius-Bode law (TBL) establishes the regularity (near-commensurability) of orbital sizes or periods in the Solar System and in exoplanet systems. TBL has not been explained convincingly yet, but the correlations between the parameters of TBL and the parameters of the orbital system revealed in this paper may be useful.

Since the TBL model depends on the preliminary numbering of the orbits, we created an improved method for objective numbering and building of an optimal TBL model. The method tests numerous possible gradients in the logarithmic version of the TBL model. It produces reasonable error curves with minima, corresponding to "good" numberings.

The method is applied to 30 orbital systems, including 17 exoplanet systems (containing at least 4 exoplanets with known masses), 2 versions of the Solar System with 6 or 8 regular planets, as well as 2 versions of the Solar System with 4 only Terrestrial Planets and with only 4 Jovian Planets, 5 systems of regular moons of the Jovian Planets plus Pluto, as well as 4 systems of small internal satellites of the Jovian Planets. We show that usually the optimal numbering (and the optimal TBL model) is not unique. For this reason in the majority of cases we explore two TBL models - main and alternative.

In the Solar Systems the rotation period of the central body supports approximatively the TBL model. However, among 8 exoplanet systems with available rotation period for the star, this rotational period is arbitrary high and useless for the fit of the TBL model. For this reason we do not use the rotational period of the central body in the case of the Solar System, too. Otherwise, from the point of view of the TBL, in comparison with other similar stars, the Sun seems to be very slow rotator.

In this paper we compare two geometric parameters of the TBL model, gradient and separability, with three physical parameters of the orbital system - mass of the central body, total mass of the orbiting bodies and (for planetary systems only) metallicity of the star.

All 10 mutual correlations between the used 5 parameters (for 18 planetary systems) occur positive. On the base of the Pearson correlation coefficient and the Student slope criterion some of these correlations may by considered as dependences. The dependence between the gradient and separability of the TBL model is the most remarkable. Generally, the gradient and the separability of the TBL model depend mainly on the total mass of the orbiting bodies, but this mass in the exoplanet systems correlates well with the metallicity of the star.

Other 6 correlations, based on the satellite systems of the solar planets, extended by the exoplanet systems, are shown. The most remarkable are the the dependences of the TBL gradient on the mass of the central body and on the total mass of the orbiting bodies The First of them seems to be linear over 8 magnitudes of the masses of the central body. The second of them is fitted by 3-rd order polynomial over 10 magnitudes of the masses of the orbiting bodies.

Harmonic resonances of the orbital periods are not discussed here.
Key words: Solar system - Titius-Bode law; Exoplanets - Titius-Bode law

Introduction

The Titius-Bode rule has been found by Johan Titius in 1766 [Wittenberg] and advertised away by Johan Bode after 1772 [Berlin]. The Titius-Bode law (TBL) has been established as a generalized heir to the Titius-Bode rule in 20-th century (Hogg 1948, Roy & Ovenden 1954, Goldreich 1965, Dermott 1968, Nieto 1972).

While the Titius-Bode rule takes place only for 8 orbits in the Solar System (the orbits of Mercury, Venus, Earth, Mars, Ceres, Jupiter, Saturn and Uranus), the TBL occurs well-performed for the solar planets, for the moons of Jupiter, Saturn and Uranus (Dermott 1968), as well for the moons of Neptune and Pluto (Georgiev 2016). Recently the validity of the TBL has been established in all, more than 200 tested exoplanet systems (Chang 2010, Bovaird & Lineweaver 2013, Huang & Bakos 2014, Bovaird et al. 2015, Altaie 2016, Aschwanden & Scholkmann 2017, Aschwanden 2018). Today any case of not-performance of the TBL would be regarded as unexpected and important news. For this reason today the interest in the TBL is increasing.

In the present paper we distinct Titius-Bode *law*, TBL, as a natural law, from Titius-Bode *relation*, TBR (but not Titius-Bode *rule*), as a TBL model, based on concrete data.

The existence of TBL as a structural law in orbital systems is not explained conventionally (Hills 1970, Hayes & Tremaine 1998, Linch 2003, Neslušan 2004). It seems harmonic resonances of orbits may be one of the key approaches (Aschwanden & Scholkmann 2017). Therefore, possible correlations between the geometrical parameters of the TBL and the physical parameters of the orbital system may be of interest for the understanding of the TBL. In our previous works we regarded 6 orbital systems in the Solar System and we found correlations between the gradient of the TBL and the mass of the central body or the total mass of the orbiting bodies (Georgiev 2016, 2017). However, the number of used orbital systems, only 6, is obviously small for final conclusions. Exploring published data (Section 1) and suitable methodics (Section 2, 3, Fig. 1, Fig. 2), here we look at TBRs for 30 orbital systems (Section 4, Appendix A) and show 16 correlations (Section 5m Appendix B).

According to the 3-rd Kepler's law the major orbital semi-axis A (in [AU]) and the relevant orbital period P (in [yr]) are connected analytically: $A \propto P^{2/3}$. Then the sense of the TBL consists of approximative regularity (near-commensurability), in which A or P grow up with acceleration while the distance from the central body increases.

In principle the TBL concerns the regular orbiting bodies, that have relatively large sizes and masses, almost circular orbits and almost complanar orbits (Dermott 1968). However, other smaller bodies in the systems often follow the TBR too (Dermott 1968, Georgiev 2016).

Usually the TBL model is presented by a power-law function

$$(1) \qquad A_n = A_0.A_C^n \qquad \text{or} \qquad P_n = P_0.P_C^n.$$

Here n = 1,2, ..., N are the numbers of the orbits or periods, A_n or P_n is the n-th major semi-axis or orbital period. The constants A_0 or P_0, as well as A_C or P_C, are considered intrinsic characteristics for every orbital system, which ought to be estimated empirically.

The model of the TBL may be presented and used also through exponential function (Poveda & Lara 2008, Panov 2009).

The constant A_C or P_C is the regularity (near-commensurability) parameter. The constant A_0 or P_0 is the scaling parameter which may be associated with orbit or period under number $n = 0$. Remarkable fact is that P_0 corresponds well to the rotation period of the central body in the system of Solar Planets, as well in systems of the regular moons of Jupiter, Saturn, Uranus (Dermot 1968), Neptune and Pluto (Georgiev 2016). For this reason the TBL is used preferably written for the orbital periods P. Another reason is that the constant P_C has resonance sense. For all solar planets $P_C \approx 2.6 \approx 5/2$. For the Terrestrial 4 Planets only and for the regular 4 moons of Jupiter $P_C \approx 2/1$.

In the logarithmic space the conventional TBL model (from Dermott, 1968, to Bovaird et al., 2015), used also in this paper, takes the form

$$(2) \qquad \log P_n = \log P_0 + G.n$$

For the sake of convenience hereafter we note the gradient (the slope coefficient) of the TBL model (Eq.2) by G:

$$(3) \qquad G = \log P_C$$

Note that the value of the constant P_C (Eq.1) does not depend on the kind of logarithm, but the gradient G (Eq.3) does.

The gradient G is our first geometrical parameter of the TBL model. One example is a reduced Solar System consisting of 4 Terrestrial planets only (#19 in Table 2 and Appendix A) with $G_1 = 0.29$ ($P_c = 1.95$, $A_c = 1.56$). Another example is a Solar System consisting of 4 Jovian Planets only (#19 in Table 2 and Appendix A) with $G_1 = 0.38$ ($P_c = 2.41$, $A_c = 1.80$).

Any TBR, based on the TBL model (Eq.2), depends crucially on the preliminary numbering of the periods (orbits). Often the numbering is not obvious. Such case is the Neptune system (#24 in Table 2 and Appendix A). For this reason we created a computer program that proposes optimal numberings for accurate TBRs (Section 2).

Occasionally the program for optimal numbering assigns one number to two periods or it reveals holes (spaces, empty numbers of periods). Thus using N input available periods the program may reveal $L \neq N$ output optimal numbers (periods). For this reason we introduce and use also a parameter, that characterizes the separability of the optimal numbering:

$$(4) \qquad S = \log L/N.$$

The separability S or L/N is our second geometrical parameter of the TBL model. One example are the exoplanets in the system of Kepler 11 (#4 in Table 2 and Appendix A) with separability $L/N = 4/6$. Another example is the regular moons in the system of Neptune (#24 in Table 2 and Appendix A) with separability $L/N = 8/3$.

Despite possible incompleteness of the lists of the known exoplanets, the separability parameter S occur useful when comparing the orbital systems.

We estimate the constants $\log P_0$ and $G = \log P_C$, plus their standard errors, together with the standard error (sample mean square deviation) σ of the TBR (Eq.2) by a fit (linear regression). For the sake of convenience we express the standard error of the TBR fit through the relative value, r, in percents:

$$r[\%] = (10^{\sigma} - 1).100 \qquad (5)$$

The method of objective numbering, presented in Section 2, shows that usually the system of the optimal numbers is not unique. For this reason we are forced to regard in this paper at least two systems of numbers (and two TBRs) – main and alternative. Two examples are shown and discussed in details in Section 3.

TBRs for 30 orbital systems are regarded in the present work, including 17 exoplanet systems with known masses of the exoplanets, 2 versions of the Solar System with 6 or 8 regular planets, 2 versions of the Solar System with 4 Terrestrial Planets only or with 4 Jovian Planets only, 5 systems of regular moons of the Jovian Planets plus Pluto, as well as 4 systems of small internal satellites of the Jovian Planets (Section 4, #0–#29 in Table 2, Table 3, Fig. 1, Fig. 2 and and Appendix A).

In this paper we reveal correlations between the geometrical parameters of the TBR – gradient G and separability S and the physical parameters of the fundamental parameters of the orbital system – mass of the central body, log M_0, total mass of the orbiting bodies, log M_S and metallicity [Fe/H] of the star (for planetary systems only). After comparing 18 planetary systems we expand the correlation ranges to include the systems of the regular moons in the Solar System, as well the systems of the small inner satellites of the Jovian Planets (Section 5, Appendix B).

The further text is divided into 5 sections. Section 1 represents the input data. Section 2 is concentrated on the method of objective numbering of the orbital periods. Section 3 introduces main and alternative TBRs on 2 characteristic examples. Section 4 represents TBRs in 30 cases. Section 5 represents mutual correlations between the regarded parameters. Section 6 summarizes the main results.

1. Input data

According to the catalog of Schneider (2017) among 616 known multiplanetary systems there are 71 systems with at least 4 exoplanets. We find and use only 17 such systems with estimated masses of the exoplanets. The Solar System with 8 regular planets is added and used as 18-th system.

Table 1 contains the input data about the planetary systems – serial number of the system, used also in Appendix A, name of the star, spectral class and metallicity [Fe/H] of the star, mass of the star, M_0, in solar masses, mass of the star, log M_0, in Earth masses, total mass of the exoplanets, log M_S, in Earth masses, number of the used planets N, and literature sources.

The available data, collected in Table 1, has variable accuracy.

The metallicity [Fe/H] and the mass of the stars M_0 are given within accuracy of 5-10 %, but for the mass of the star HR 8799 the accuracy is about 20 %. Error estimations are not given for the masses of the stars Gliese-876, Kepler-89 and μ Arae.

The masses of the exoplanets are estimated with low accuracy, 10-40 %, but for the system Kepler-62 the errors seem to be about 100 %. Error estimations are not given for the exoplanets around the stars Kepler-80, Kepler-107 and μ Arae. Only lower limits of the masses of the explanets are estimated for the systems in Table 1 with numbers from #10 to #17. However, the data about these systems do not deviate remarkably from the correlations and we considered these data may be used too.

Some other remarks also are necessary. In the case of Gliese-876 the exoplanets 2 and 3 (f and g) are used here as "confirmed". In the case of Kepler-11, after comparison of the radii of the other exoplanets, the mass of the most distant and unconfirmed exoplanet (g) is adopted to be 8 Earth masses. In the case of Kepler-80 the mass of the most inner exoplanet (f) was adopted to be 3 Earth mases. At Gliese-581 the exoplanets 4 and 5 (g and d) are used as "confirmed". In the case of HD 10180 the exoplanets 3 and 6 (i and j) are used as known too.

Table 1. Input data about the regarded planetary systems(see the text)

#	Star	Class	[Fe/H]	$M_0[M_\odot]$	$\log M_0[M_\oplus]$	$\log M_S[M_\oplus]$	N	Source
1	Gliese 876	M4V	0.19	0.37	5.091	3.017	6	[R2010]
2	HR 8799	A5	0.20	1.47	5.690	3.917	4	[M2010]
3	HR 8832	K3	0.20	0.79	5.422	2.204	7	[N2015]
4	Kepler-11	G6V	0.00	0.96	5.505	1.479	6	[L2013]
5	Kepler-20	G8V	0.02	0.91	5.482	1.784	6	[F2011],[B2016]
6	Kepler-80	M0V	-0.56	0.73	5.386	1.440	5	[M2016]
7	Kepler-89	F8V	-0.01	1.25	5.619	2.125	4	[T2013],[M2013]
8	Kepler-107	G2V	0.09	1.18	5.594	1.313	4	[EDE2017]
9	TRAPPIST-1	M8V	0.04	0.08	4.427	0.602	7	[H2016],[G2017]
10	Ups And A	F8V	0.08	1.27	5.626	3.914	4	[W2009]
11	55 Cns A	G8V	0.21	0.95	5.500	1.134	5	[D2010],[W2011]
12	Gliese-581	M3V	-0.33	0.31	5.014	1.508	5	[R2014]
13	Gliese-676	M0V	0.23	0.71	5.374	3.971	4	[A2012]
14	HD 10180	G1V	0.08	1.06	5.549	2.229	9	[T2012]
15	HD 40307	K2V	-0.31	0.75	5.397	1.555	6	[TA2012]
16	Kepler-62	K2-5V	-0.37	0.69	5.361	1.176	5	[B2013]
17	Mu Arae	G3IV	0.30	1.10	5.564	3.109	4	[P2006]
18	Solar System	G2V	0.00	1.00	5.522	2.650	8	[IAU2006]

Generally, the available data on multiplanetary systems are not too full and too accurate. Though, they occur good enough for the purposes of this work.

2. Objective numbering of the orbital periods for TBR

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Each TBR is based on a preliminary numbering of the periods (orbits). Often the numbering is not obvious nor unique. For this reason an objective method for numbering is used in the present work. It is an improved version of the method, used in the paper of Georgiev (2016). We created a computer program that scans reasonable interval of TBR gradients G (Eq. 3). Usually we use a testing interval for G from 0.1 to 0.7 and scamming step of 0.001. The program produces error curves whose local minima corresponds to "good" numberings, as follows.

Initially, the input numbers of the available orbital periods in increasing order are P_k, k = 1, 2, ..., N. Further N-1 differences Q_k = log P_k - log P_1, k = 2, 3 ,..., N, are explored. For each tested value of G the program derives N-1 quotients $u_k = Q_k/G$. It finds also the integer m_k that is the nearest to u_k and derives the relevant error difference $e_k = u_k - m_k$. The mean square value of all such differences is used as an error function ε in dependence on G:

$$(6) \qquad \varepsilon = [\Sigma e_k^2/(N-2)]^{1/2}, \quad k = 2,\ 3\ ,...,\ N.$$

This error function characterizes the goodness of the numbering m_k, k = 1, 2, ..., N, corresponding to the tested value of G. The behavior of ε in dependence on G is an almost smooth curve, whose minima correspond to "good" numberings. Such error curves, transformed in relative values, r[%] (Eq.5), are shown in the low-right corners of TBR diagrams in Fig. 1, Fig. 2 and Appendix A.

The program applies the numbers m_k, corresponding to each tested value of G, to also fit a TBL model (Eq.2). The individual deviations from the fit are d_k = log P_n - log P_0 - $G.n_k$. Then the fit standard error, which may also be regarded as an error function on G, is

$$(7) \qquad \sigma = [\Sigma d_k^2/(N-2)]^{1/2}, \quad k = 2,\ 3\ ,...,\ N.$$

Such error function is presented by steps in the low-right corners of the TBR diagrams in Fig. 1 and Fig. 2 only. In the bounds of each step of this function different values of G produce the same series of numbers.

Furthermore, the user chooses from a minimum of the error curve (Eq.6) an approximate value of a "good" gradient and introduces it in the same program. In a second run the program derives accurate statistics of the TBR fit under the chosen "optimal" numbering. (In the first run the program the introduced gradient is dummy, but belonging to the chosen testing interval for G.)

The output numbers of the orbital periods, are m_k, k = 1, 2, ..., N. Occasionally $m_k \neq n_k$ and the full output number of periods is $m_N = L$. Usually $L \neq N$. Then the separability quotient L/S (Eq.4) characterizes the optimal rarefaction of the periods (orbits). If the user chooses another "optimal" gradient, he derives another "optimal" TBR (Section 3).

By default the first input number is $n_1 = 1$ and the first output number is $m_1 = 1$. But the gradient of the TBR numbering is invariant in respect to an additive integer to the numbering. Therefore, the use of another first input/output number is admissible. In 6 TBRs in the Solar System the rotational period of the central body supports the TBR and it is used for the fit under number $n_1 = m_1 = 0$ (Dermott 1968, Georgiev 2016).

However, in the available exoplanet systems the stellar rotational period occurs typically high and unusable for the fit of the TBR (Section 4.1). For this reason we do not use the rotational period of the central body in all TBR fits in the present work. Though, sometimes we add reasonable integer number for TBR in the Solar System to check how much the rotation period of the central body corresponds to $m_1 = 0$ (Section 4.2).

Furthermore, the output numbers of the orbital periods in the TBR diagrams, signed along the abscissa axes of the diagrams, are noted as usual by n.

3. Main and alternative TBRs with 2 characteristic examples

Usually the error curve (Eq. 6), produced by the method for objective numbering (Section 2), shows a few local minima and so the choice of the really optimal TBR numbering is difficult. Here we regard 2 TBRs, based on 2 "good" numberings - main and alternative. Characteristic examples about the planetary systems of the Sun and 55 Cns follows.

Figure 1 shows error curves (Eq.6,7) and TBRs in an artificially simplified version of Solar System with 6 planets. The Earth and Neptune are excluded because of their significant deviations from the "standard" TBR (Dermott 1968, Neslušan 2004). Thus the error curve (Eq. 6), shown in the low-right corner, becomes simple and clear. The wide and deep main minimum gives the optimal gradient $G_1 \approx 0.42$, corresponding to P_C =2.64 and $A_C = 1.91$. The respective optimal numbers are signed along the left solid regression line in Fig. 1. This is the "main" TBR with standard error $r = 6.6$ % (Eq.5).

In Fig. 1 the orbital periods of the planets, used for the fit, are presented by dots. The fit predictions are marked by large circles. The main TBR, presented by solid line, is very close to the "standard TBR", where one hole under $n = 4$, corresponds to the Main Asteroid Belt (or to Ceres). The optimal position of the Earth, together with Venus ($n = 2$), Ceres ($n = 4$), Neptune, together with Pluto ($n = 8$), Eris ($n = 9$), as well as the rotation period of the Sun ($P_S = 25$ days, $n = 0$), are marked by small circles. The rotation period of the Sun, as well as the orbital periods of the Earth and Neptune, show the largest deviations from the fit. The gradient (Eq.3) and separability (Eq.4) of this main TBR are $G_1 = 0.42$ and $L/N = 7/6$.

In Fig. 1 other deep but narrow minima of the error curve (Eq.6) correspond reasonably to $G_2 = G_/2$ and to $G_3 = G_1/3$. This special example is created mainly to show well these "harmonic" minimums. In such clear case the depth of these minima are almost the same as the depth of the main minimum, but usually the main minimum is more shallow. Here the appropriated "alternative" TBR, corresponding to G_2, is shown by the right dashed regression line. The parameters of the main and alternative TBRs are included in Table 2 under #0, but they are not used for the correlations in Section 5.1.

In the alternative TBR the optimal numbers, beginning by default by $n = 1$, are increased additionally (artificially) by 2. Thus, the rotation period of the Sun is well predicted under number $n = 0$. In this TBR Mercury takes number 3, while the numbers 1 and 2 occur empty. The

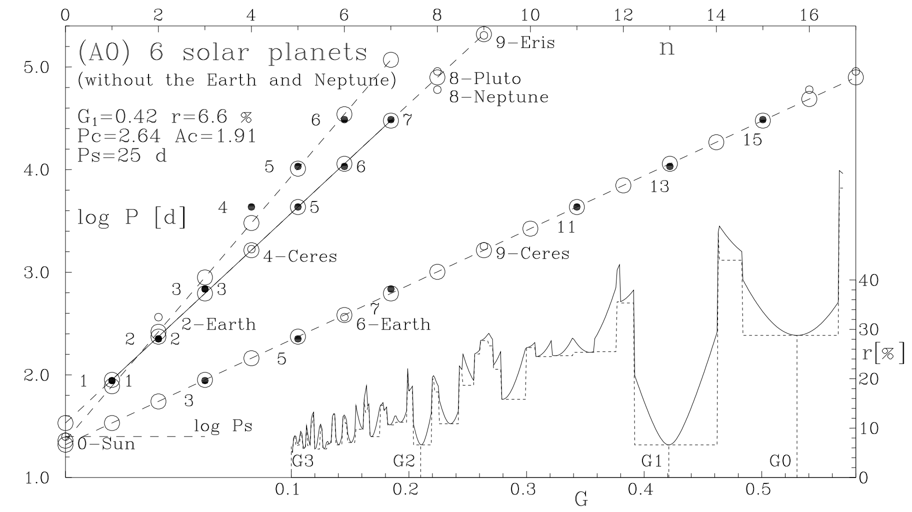

Fig. 1. Error curves (Eq.6, 7) in the low-right corner and TBRs regression lines, corresponding to gradients G_0, G_1 and G_2, for a simplified Solar System with 6 planets. Upper abscissa axis contains output TBR numbers, n. Ordinate axis corresponds to orbital periods, log P, in days. Dots represent data, used for the fits. Circles represent the fit predictions. Small circles show the best positions of other orbiting bodies and the rotation period of the Sun. Horizontal dashed line is the level of the rotation period of the Sun. The parameters of the main TBR (solid line) are noted in the diagram. (See also the text.)

Earth falls into number 6 and does not appear as extraordinary planet in the system. Totally 7 numbers occur free between Mercury and Uranus. There the Earth, Ceres, Neptune, Pluto and Eris take the predicted numbers 6, 9, 17, 18 and 20, respectively. The gradient of the alternative TBR is $G_2 \approx 0.21$. Since the numbers of Mercury and Uranus are 3 and 16, the separability (Eq.4) is $L/N = 13/6$.

Which TBR version in Fig. 1 is better, the main or the alternative? In this simple case the alternative TBR (i) has gradient $G_2 = G_1/2$ and (ii) it poses the same accuracy as the main TBR. Therefore, this alternative TBR, shown by the right dashed line, should be ignored. However, usually the conditions (i) and (ii) are not fulfilled, even the alternative TBR is significantly more accurate. For this reason in the cases of the planetary systems (Diagrams #1-#20, Appendix A) we regard two TBRs - main, corresponding to G_1, and alternative, corresponding to G_2. We regard main and alternative TBRs even in cases with $G_2 = G_1/2$. However, for the satellite systems of the Solar planets, excluding regular moons of Saturn, (Diagrams #21-#25, Appendix A) the method of the objective numbering (Section 2) reveals only one, main, TBR.

In Fig. 1 the most right part of the error curves (Eq.6) shows wide and shallow minimum, centering on the gradient $G_0 \approx 0.56$. In the respective TBR, presented by the left dashed line, the Earth falls again on $n = 2$, together with Venus, but the hole of the Main Asteroid Belt is absent. This

TBR, having very high standard error, ≈ 30 %, seems to be too rough. Hereafter, such rough TBRs are ignored.

Figure 2 presents the complicated case of the exoplanet system around the star 55 Cns (#11 in Table 2). This is a detailed version of Diagram A11, pulled out from Appendix A. The TBR, corresponding to $G_0 \approx 0.66$ ($P_C = 4.57$, $A_C = 2.75$), with $r \approx 20$ % is presented by a dashed line. This is the most steep TBR among all TBRs in this paper. One similar TBR, presented by the most left short dashed line, is found by Poveda & Lara (2008). These authors predict the periods of 2 unknown exoplanets (open squares), one internal and one external for the known system. The shift between these TBRs is due to different values used for the first orbital period. But if we add 1 to the numbers of Poveda & Lara (2008), both TBRs coincide. Using more accurate input data (#11 in Table 1) we may predict through our TBR fit (dashed line) two internal planets, marked by open circles. Though, in the present paper we ignore this rough and extraordinary steep TBR.

In Fig. 2 we consider that the main TBR of 55 Cns is characterized by $G_1 = 0.43$ and $r = 9.2$ %. It is shown by left solid curve. Yjis TBR reveals 5 internal free numbers, which may correspond to unknown planets. The total output number is $L = 10$ and the separability is $L/N = 10/5$. Almost the same TBL model is found by Bovaird & Lineweavwe (2013), noted by right short dashed line. These authors predict 3 unknown exoplanets, 2 internals and 1 external (open squares). Both TBRs are sightly distinct because of slightly different input data. The TBR, found by Curtz (2012), not shown here, predicting 4 unknown internal exoplanets, practically coincides with our main TBR.

In Fig. 2 the alternative TBR (right solid line) has $G_2 = 0.26$, $r = 3.5$ % and $L/N = 16/5$ (#11 in Table 2). In contrast to the Solar System #0, (i) the gradient of the alternative TBR of 55 Cns is not harmonic of the gradient of the main TBR and (ii) the accuracy of the alternative TBR is significantly higher. We can not ignore this alternative TBR. Moreover, such pairs of TBRs dominate among the exoplanet systems.

After excluding of the most rough TBR (dashed line) the main and alternative TBRs of 55 Cns occurs very similar to the main TBR of the Solar System (# 18 in Table 2). However, the known size of the system of 55 Cns seems to be about 10 times shorter in comparison with the Solar System including Neptune. The majority of the exoplanet system have such short sizes.

Besides, while the rotation period of the Sun ($P_s = 25$ days) supports the TBL models of the Solar System under number 0 (#0, #18-#20), the rotational period of 55 Cns ($P_s = 42$ days) corresponds well with the orbital period of the 3rd known exoplanet there.

4. TBRs for 30 orbital systems

The method for deriving the main and the alternative TBRs (Sections 2, 3, Fig. 1, 2) is applied to 30 orbital systems. The results are presented in Table 2, Table 3, and Appendix A. The diagram A11, concerning the system 55 Cns, is shown on Fig. 2.

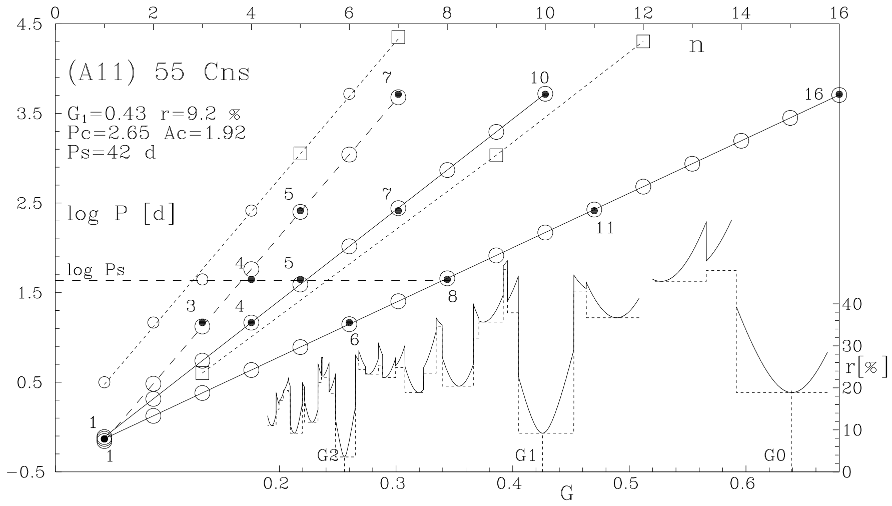

Fig. 2. Error functions and TBRs for the complicated system 55 Cns. Solid lines correspond to the main and alternative TBRs. Dashed line shows the rough TBR, whith ia ignored. Short dashed lines and squares show TBRs and predicted planets of other authors. (See Fig. 1 and the text.

Diagrams A1-A20 and Table 2 present 17 exoplanet systems (#1-#17), Solar System with 8 planets (#18), Solar System with 4 Terrestrial Planets only (#19) and Solar System with 4 Jovian Planets only (#20). Diagrams A21-A25 and Table 3 present 9 satellite systems in the Solar System (#21-#29). Juxtapositions of the parameters of the orbital systems are presented in Section 5 and Appendix B as diagrams B1-B16.

Solar System with 4 Terrestrial Planets only is considered for comparison with the known parts of the exoplanet systems (B1-B10). These parts are typically not large and not multinumerous. Solar System with 4 Jovian Planets only is intended for comparison with the systems of regular moons of the solar planets (B11-B16). Since the total mass of the Jovian Planets exceeds the total mass of the Terrestrial Planets about 250 times, the Jovian Planets are just the regular bodies in the Solar system.

The right bottom corners of the diagrams A1-A25 show error functions ε (Eq.6), transformed to relative values r % (Eq.5), in dependence on the tested gradients G (Eq.3). In the case of the satellites of the Jovian Planets (A21-A24) two error functions are presented, one for the inner small satellites only (left) and another for the regular moons (right).

For the planetary systems and the regular moons of Saturn the positions of the minima G_1 and G_2 are used for deriving the main and alternative TBRs. Alternative TBRs are not found for the regular moons of Jupiter, Uranus, Neptune and Pluto, as well as for the small internal satellites of Jupiter, Saturn, Uranus and Neptune. For the inner satellites of the last mentioned 4 planets the positions of the minima are marked by G_i (A21-A24).

The upper parts of the diagrams in Appendix A present TBRs. The abscissa axis contains the output system of the period numbers n (Sections 2,3). The ordinate axis corresponds to the orbital periods, log P, in days. Dashed horizontal lines show the logarithmic values of rotation periods of the stars (if they are available), noted by P_S, and of the planets, noted by P_P, in days.

The dots in the diagrams corresponds to the periods, used for the TBR fits. Solid lines represent the fits (linear regressions) of the main and alternative TBR over the used periods. The relevant optimal (output) TBR numbers of the used periods are signed along the lines. Open circles show the fit predictions. Empty circles may be regarded as predictions for stable periods or for unknown (or not existing) orbiting bodies. In the diagrams of the Solar system (A18-A25) the positions of other known bodies are marked by small circles.

In diagrams A1-A17 for the exoplanet systems input and output numbers of the shortest periods, used for the fit, are $n = 1$. However, in diagrams A18-A25 for the Solar System the output numbers are sometimes increased additionally to ensure the position of the central body close to number $n = 0$. For example, in the case of Jupiter (A21) such goal is reached by increasing the numbers by 1. Then number $n = 1$ in this system rests empty.

Short dashed lines in the diagrams A1, A10, A15 and A17 correspond to TBRs, found by Bovair & Lineweaver (2013), where the empty squares show predicted periods of unknown exoplanets. The distinctions between the TBRs of Bovair & Lineweaver (2013) and our TBRs are due to different methods of TBR building and to small distinctions in the input data.

4.1. TBRs for 21 planetary systems

Table 2 summarizes the TBR results about 17 exoplanet systems (#1-#17) and 4 versions of the Solar system (#0, #18, #19 and #20). There the values G_1 and G_2 are the gradients (Eq.3) of the main and alternative TBRs, followed by the relevant standard errors of the gradients $\sigma(G_1)$ and $\sigma(G_2)$, relative standard errors of the TBRs $r(\text{TBR})[\%]$ (Eq.5) and the separability parameter L/N (Eq.4).

Table 2 shows that the TBRs of the exoplanet systems #1-#17 may be characterized by different relative accuracy: 8.4-28.5 % for the main TBRs and 3.1-13.3 % for the alternative TBRs.

The ranges of the gradients and separabilities of the main TBRs here are bounded by the exoplanet systems of TRAPPIST-1 and Gliese 676: $G_1 = 0.286$-0.470 ($P_C = 1.93$-2.95, $A_C = 1.55$-2.06) and $L/N = 5/7$-$12/4$. One serious exception is the system of 55 Cns, whose rough TBR with $G_0 = 0.66$ is ignored (Fig. 2).

The total number of empty periods (spaces) in the planetary sequences is 25 for the main TBRs and 98 for the alternative TBRs, while the cases when the same period is assigned to 2 periods (orbits) is respectively 7 and 1.

Solar System needs special attention.

In the version with 8 regular planets (#18) the parameters of the TBRs are very close to the parameters in the case of 6 regular planets (#0). In

Table 2. Output data about the main and alternative TBRs for planetary systems, shown in Fig. 1,Fig. 2 and Appendix A. (See the text.)

#	Star	G_1	$\sigma(G_1)$	r(TBR)[%]	S_E	G_2	$\sigma(G_2)$	r(TBR)[%]	L/N
0	Sol.Sys.[6p]	0.421	0.005	6.6	7/6	0.210	0.003	6.6	13/6
1	Gliese 876	0.294	0.011	12.6	7/6	0.151	0.002	3.8	13/6
2	HR 8799	0.332	0.011	6.0	4/4	0.166	0.006	5.9	7/4
3	HR 8832	0.403	0.010	17.7	8/7	0.279	0.004	10.2	11/7
4	Kepler-11	0.327	0.028	18.6	4/6	0.179	0.007	8.1	7/6
5	Kepler-20	0.328	0.028	24.0	5/6	0.261	0.009	9.4	6/6
6	Kepler-80	0.306	0.037	23.7	4/5	0.167	0.003	3.7	7/5
7	Kepler-89	0.383	0.017	8.4	4/4	0.192	0.008	8.4	7/4
8	Kepler-107	0.333	0.077	28.5	3/4	0.221	0.019	6.9	4/4
9	TRAPPIST-1	0.286	0.023	19.3	5/7	0.152	0.005	6.0	8/7
10	Ups And A	0.413	0.010	13.1	8/4	0.244	0.001	1.9	13/4
11	55 Cns A	0.426	0.006	9.2	10/5	0.258	0.001	3.5	16/5
12	Gliese-581	0.343	0.016	12.3	5/5	0.190	0.006	7.6	8/5
13	Gliese-676	0.476	0.018	14.4	12/4	0.300	0.006	13.3	21/4
14	HD 10180	0.330	0.009	18.8	11/9	0.221	0.003	8.1	16/9
15	HD 40307	0.330	0.016	15.4	6/6	0.195	0.004	6.1	10/6
16	Kepler-62	0.426	0.023	19.2	5/5	0.271	0.010	12.7	7/5
17	Mu Arae	0.375	0.007	8.8	8/4	0.165	0.001	3.1	17/4
18	Sol.Sys.[8p]	0.404	0.011	19.7	8/8	0.214	0.003	9.1	16/8
19	Sol.Sys.[4T]	0.289	0.028	15.4	4/4	0.221	0.011	7.6	5/4
20	Sol.Sys.[4J]	0.382	0.023	12.3	4/4	0.226	0.011	10.7	6/4

the main TBR Venus and the Earth take $n = 2$, the period $n = 4$ is empty and the rotational period of the Sun is about 1.6 times less than the TBR prediction. In the alternative TBR, Venus and the Earth occupy numbers 4 and 5, many numbers occur empty and the rotational period of the Sun is better predicted. The accuracy of the alternative TBR is 2 times higher.

The TBRs of the reduced Solar System, containing 4 Terrestrial planets only (#19) shows relatively low gradients and relatively low accuracy. The main TBR predicts well the rotational period of the Sun and an empty period under $n = 1$. However, the main TBR is not valid for more distant parts of the Solar System. The alternative TBR does not predict the rotational period of the Sun and predicts two empty periods close to the Sun, but it is valid for the periods of Ceres and Jupiter.

The TBRs of the reduced Solar System, containing 4 Jovian Planets only (#20), answers to the condition of the use of regular orbiting bodies only. These TBRs show relatively low accuracy and they form different systems of periods in the region of the Terrestrial Planets. However, the TBRs, based on the 4 Jovian Planets only, predict well the rotation period of the Sun.

Generally, from the point of view of the TBL, it seems two planetary populations cohabit in the Solar system.

In the end, among 8 exoplanet systems with known rotational period of the star (Appendix A, dashed horizontal lines) the rotational period of the star stands typically high and placed among the orbital periods of the exoplanets. The relevant stars are similar to the Sun, but from the point of view of the TBRs, the rotation of the Sun seems slow.

4.2. TBRs for 9 satellite systems of solar planets

The TBRs for about 9 satellite systems of solar planets are presented in diagrams A21-A25. The relevant input/output data are collected in Table 3 (#21-#29). In contrast to works of Dermott (1968) and Georgiev (2016, 2017), but similar to the work on exoplanet systems here we do not use the rotational period of the central body for the TBR fit.

We regard TBRs for the regular moons of the Jovian Planets plus Pluto, as well as for inner small satellites of the Jovian Planets. The sources of data are NASA (Solar system exploration, https://solarsystem.nasa.gov/) and JPL (Planetary Satellite Physical Parameters, https://ssd.jpl.nasa.gov/).

The systems of Neptune and Pluto contain only 3 satellites, which may be considered as regular. Still, these systems occur useful when comparing correlations between the orbital systems and they are explored too.

Table 3 summarizes the input and output data about the satellite systems of solar planets. The numeral columns in Table 3 contain mass of the planet, log M_0, in Earth masses, total mass of the used satellites, log M_S, in Earth masses, TBR gradient G (Eq.3), standard error of the gradient σ_G, relative standard error of the TBL model r(TBR)[%] and separability parameter (Eq.4)of the TBR L/N.

Table 3. Input and output data about systems of moons (#21-#25]; 21-A25) and systems of small inner satellites of the Jovian Planets (#26-#29); A21-A24). (See the text.)

#	Name	log$M_0[M_\oplus]$	log$M_S[M_\oplus]$	G(TBR)	σ_G(YBR)	r(TBL)[%]	L/N
21	Jupiter	2.50	-1.14	0.323	0.011	5.92	4/4
22	Saturn	1.95	-1.63	0.243	0.007	11.90	9/7
22	Saturn-2	1.95	-1.64	0.184	0.002	6.27	13/7
23	Uranus	1.17	-2.81	0.250	0.010	7.62	5/5
24	Neptune	1.24	-2.44	0.358	0.001	0.55	8/3
25	Pluto	-2.66	-3.58	0.195	0.002	1.37	5/3
26	Jupiter-i	2.50	-6.40	0.116	0.003	1.79	4/4
27	Saturn-i	1.95	-6.34	0.053	0.007	1.80	2/5
28	Uranus-i	1.17	-6.06	0.071	0.001	4.42	6/10
29	Neptune-i	1.24	-5.72	0.056	0.001	5.18	5/6

In Table 3 the string "Saturn2" contains data about the alternative TBR of the system of regular moons of Saturn. The last 4 strings contain data about the system of small inner satellites of the Jovian Planets.

The satellite system of Jupiter (Sheppard 2016) is dominated by the 4 Galilean moons (Jo, Evropa, Ganimede, Calisto). We shifted additionally their output TBR numbers by 1 and the result numbers become respectively 2, 3, 4, 5. Thus the rotational period of Jupiter supports well the TBR under

$n = 0$, while the orbital period $n = 1$ is empty (#21, A21). We build TBR also for 4 small inner satellites with diameters 15-167 km. Their output optimal TBR numbers are Metis – 1, Adraste – 1, Amalthea – 3 and Thebe – 4 (#26, A21).

The moon system of Saturn (Jacobson et al. 2006) occurs very complicated and we are forced to regard main and alternative TBRs (#22, A22). The output numbers for the 7 regular moons (Mimas, Enceladus, Tethis, Dione, Rhea, Titan, Iapetus) in the main TBR (without shift) are 1, 2, 2, 3, 4, 6, 9, respectively. The alternative TBR (after shifting by 1) is more rarefied but more accurate. The rotational period of Saturn does not support well any TBR. Six inner small satellites, with diameters 15-179 km, support another TBR. They take only 2 different optimal TBR numbers: Pan – 1, Atlas –1, Prometheus – 1, Pandora – 2, Epometheus – 2, Janus – 2 (#27, A22).

In the system of Uranus (Jacobson et al. 1992) 5 regular moons (Miranda, Ariel, Umbriel, Titania, Oberon) form a TBR which is well supported by the rotational period of Uranus (#23, A23). Ten inner satellites, with diameters 40-160 km, support another TBR under 5 different optimal TBR numbers: Cordelia – 1, Ophelia – 2, Bianca – 3, Cressida – 3, Desdemona – 3, Juliet – 3, Portia – 3, Rosalind – 4, Belinda – 4, Puck – 5 (#28, A23).

The system of Neptune (Jacobson 2009) is strongly rarefied. It consists of only 3 regular moons (Proteus, Triton, Nereid). They take output numbers 1, 3 and 8, with separability $L/N = 8/3$. The rotation period of Neptune does not support well the TBL model (#24, A24). Six inner small satellites with diameters 16-194 km support their own TBR with 5 different optimal TBR numbers Naiad – 1, Thalassa – 1, Despina – 1, Galatea – 2, Larissa – 3 and S/2004 – 5 (#29, A24).

At the end, the system of Pluto (Brozovic 2015) contains only 3 satellites which may be considered regular (Charon, Nix, Hydra). They support a rough TBR under numbers 1, 3, and 4, shown in A25 by dashed line and ignored. The adopted here accurate TBR assigns satellite numbers 1, 4, and 5, respectively (#25, A25). (The rotational period of Pluto and the orbital period of the closest satellite Charon are synchronized.)

The TBRs of the moon systems of the solar planets (Table 3) occur typically more accurate than the main TBRs of the planetary systems (Table 2). Only the main TBR for the moons of Saturn within standard error 11.9 % is relatively rough.

The TBRs of the small satellites are well pronounced, but their gradients are 2-4 times less then the gradients of the TBRs for the regular moons.

5. Correlations between TBR parameters and physical parameters of orbital systems

Hereafter we compare the geometrical parameters of the TBRs – gradient $G = \log P_C$ (Eq.3) and separability $S = \log L/N$ (Eq.4), with the physical parameters of the orbital systems – mass of the central body, $\log M_0$, total mass of the orbiting bodies, log M_S and, for planetary systems only, metallicity of the star [Fe/H]. The sources of data are Table 2 and Table 3.

Appendix B contains diagrams B1-B10, which juxtapose parameters of the planetary systems only (#1-#18), as well as diagrams B11-B16 which juxtapose the parameters of satellite systems, together with the parameters of planetary systems. Diagrams B9 and B10 are shown in large format as Fig. 4 and Fig. 5. Diagrams B15 and B16 are shown in large format as Fig. 6 and Fig. 7.

The points in the diagrams represent different data, as follows: dots (1) – main TGRs for planetary systems, circles (2) – alternative TBRs for planetary systems, filled squares (3) – main TBR for Solar System with 4 Terrestrial Planets only, open squares (4) – alternative TBR for Solar System with 4 Terrestrial Planets only, filled triangles (5) – main (unique) TBRs for the regular moons of solar planets, open triangles (6) – alternative TBR for the regular moons of Saturn only, (7) – main (unique) TBRs of the inner small satellites of solar planets. The dots (1) and filled squares (3) are used also for juxtaposition of physical parameters in the diagrams B1, B3, B5 and B11 for planetary systems and Solar System with 4 Terrestrial Planets only.

The parameters of the main TBRs show better pronounced correlations than the parameters of the alternative TBRs and the last mentioned are not especially commented furthermore. Error bars of the gradients of the main TBRs are shown in the diagram B2 for planetary systems, in B15 (Fig. 6) for regular moons and B16 (Fig. 7) for inner small satellites.

5.1. Correlations for 18 planetary systems

Diagrams B1-B10 represent correlations for 17 exoplanet systems (#1-#17) plus Solar System with 8 planets (#18). The numbers of the points correspond to the numbers of the orbital systems in Table 2. The solid lines represents fits over the data, shown by dots, while the dashed lines represent fits over the data, shown by circles. Solar System with 4 Terrestrial planets only (#19) is not used for the fits. The correlations are characterized in Table 4 and compared in Fig. 3.

We concentrate on the correlations for the planetary systems, based on the parameters of their main TBRs. The fits (linear regressions) in the diagrams have the common form

$$y = y_0 + B.x \tag{8}$$

with standard error of the regression σ_y and standard error of the slope coefficient σ_B.

The significance of the slope coefficient B is characterized by the Student test parameter T:

$$T = |B|/\sigma_B. \tag{9}$$

Large value of T corresponds to statistically significant difference between —B— and 0. For our 18 points the 99 % confidence level is overcome by $T > 0.95$.

Table 4. Table 4. Statistical parameters of the solid regression lines of the correlations in diagrams B1-B10 for the planetary systems. (See the text)

#	Parameterss	Diagram	σ_y	C	B	B/σ_B
1	$M_0 - M_S$	B1	0.967	0.431	1.504	1.91
2	$M_S - G$	B2	0.051	0.363	0.018	1.56
3	[Fe/H]-M_S	B3	0.921	0.512	2.217	2.39
4	[Fe/H]-G	B4	0.052	0.237	0.052	0.98
5	[Fe/H]-M_0	B5	0.301	0.186	0.231	0.76
6	[Fe/H]-S	B6	0.170	0.479	0.358	2.18
7	$M_0 - G$	B7	0.050	0.384	0.068	1.66
8	$M_0 - S$	B8	0.032	0.250	0.151	1.03
9	$S - G$	B9	0.038	0.724	0.214	4.20
10	$M_S - S$	B10	0.250	0.623	0.107	3.18

The closeness between the general behavior of x and y values is characterized generally by the Pearson's correlation coefficient C. If —C— is close to 1, the correlation may be considered as dependence.

Table 4 contains the basic parameters of the fits for the planetary systems, concerning the main TBLs only: standard error of the fit σ_y, correlation coefficient C, slope coefficient of the fit B and test parameter B/σ_B (Eq.9).

Figure 3 represents comparison between the values of C and T, collected in Table 4. It elucidates at least two important particularities of the regarded correlations.

First, all correlations in the diagrams B1-B10 are positive. The TBR parameters G and S increase with the increase of the physical parameters log M_0, log M_S, and [Fe/H]. Also, the values of log M_0 and log M_S increase with the increase of [Fe/H]. By these mutual correlations, represented as planes, f.e. $G = F($[Fe/H], log $M_S)$, tested by us, have not significantly lower standard errors in comparison with the linear fits, f.e. $G = F($log $M_S)$ (Diagram B2).

Second, because of their high values of T the majority of the correlations may be considered as dependences. However, while the slope parameter T overcomes significantly the level of 99 % confidence probability, the respective correlation coefficient C is relatively low for the majority of the correlations. The imperfect data about the exoplanet systems influence surely, but the main reason for this discrepancy seems to be other. In statistics the parameters T and C are defined for normally distributed random variables. However, our 5 regarded parameters, as well as the residual deviations from the fits, have nearly flat distributions. For these reasons the values of C becomes relatively low, while the values of T become relatively high. Therefore we may call "dependences" only well-pronounced correlations, such as $S - B$ (B9; Fig. 4) and log $M_S - S$ (B10; Fig. 5).

Figure 4 shows dependence between the geometrical parameters of the main TBRs – separability S and gradient G. Such dependence is poorly

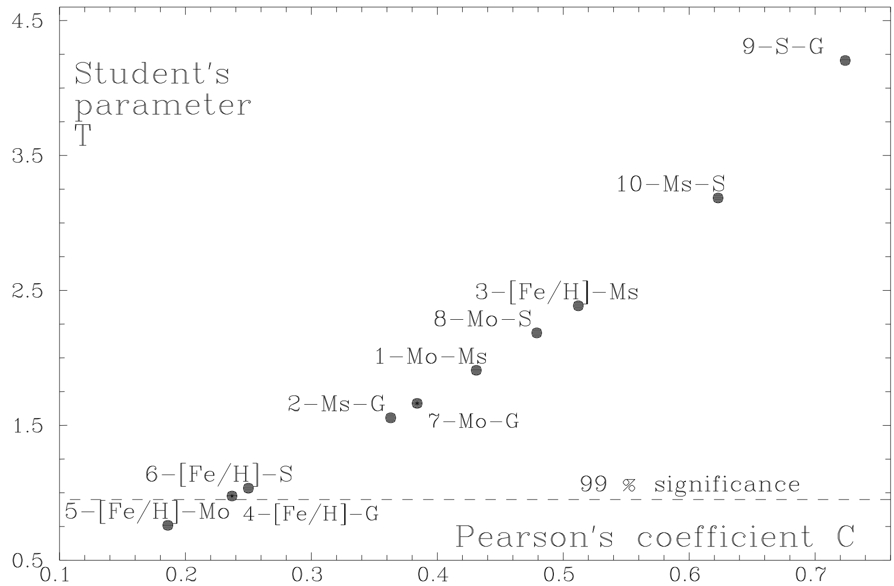

Fig. 3. Dependence between the correlation coefficient C and the significance parameter T (Eq.9) for the main TBRs of the planetary systems (Table 4). The horizontal dashed line shows the level of 99 % significance of the slope coefficient.

pronounced for the alternative TBRs. Otherwise, the gradient correlates well with log M_S and [Fe/H] (Diagram B2 and B4).

Figure 5 shows that the separability S of the main and alternative TBRs depends on the total mass of the planets, log M_S. Otherwise, the separability correlates with M_0 and [Fe/H] (Diagrams B8 and B6).

Diagram B1 and B3 show other remarkable correlations: log M_0 – log M_S and [Fe/H] - log M_S. Diagrams B4 and B6 give evidences that the metallicity [Fe/H] influences also G and S. Generally, the metallicity of the star seems to be significant parameter of the structure of the orbital system.

Diagrams B1-B10 show that a Solar System with 4 Terrestrial Planets only (#19), differs slightly from the planetary systems by gradient G and separability S, but, naturally, differs significantly by mass M_S (B1, B3). The position of such reduced Solar System gives evidence of possible deficiency of exoplanets which are distant from their stars. Practically, the discovery of such exoplanets is difficult because their gravitational influence on the star may be too faint or because their transits in front of the star may be too rare.

We note that some exoplanet systems, seeming to be unique, affect significantly the correlations. System TRAPPIST-1 (#9 in Table 2), having extremely small masses M_0 and M_S, is important in diagram B1, but it shifts the fits in diagrams B3 and B5 downwards. System Gliese-676 (#13), having extremely high separability ($L/N = 21/4$), obviously increases the regression slopes and significances of diagrams B6 and B8. Other such ex-

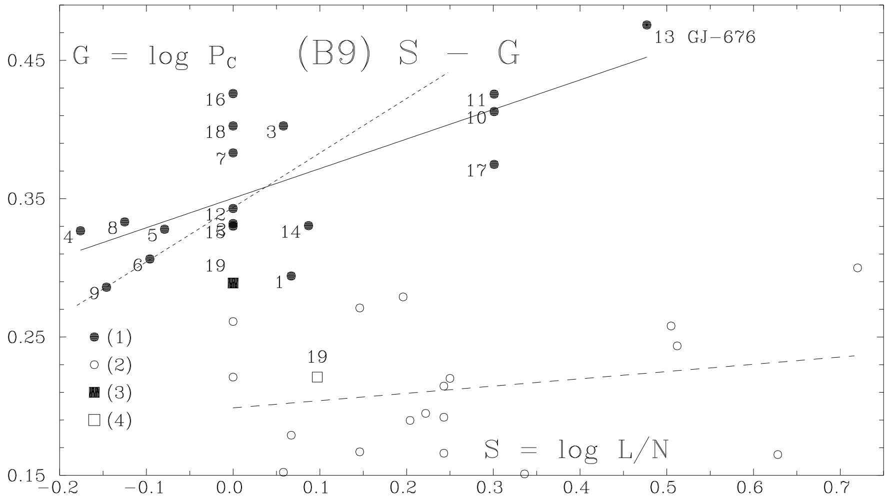

Fig. 4. Dependence between the separability parameter S and the gradient G for the main TBRs (dots, solid line; short dashed line shows the reverce regression) and correlation between S and G of the alternative TBRs (circles, dashed line). Data about 18 planetary systems are used. (See Table 4.)

amples are 55 Cns (#11) on diagrams B2 and B10 (Fig. 5), as well as Kepler-62 (#16) on diagrams B2 and B4.

Generally, it is not sure whether we have to observe more close correlations, but the choice of one optimal TBR numbering must give more sure results.

Other physical parameters, based on the masses M_i, orbital periods P_i, major semi-axes A_i and linear velocities V_i of the exoplanets, were also tested for correlations. The parameters $\Sigma M_i.A_I$ and $\Sigma M_i.P_I$, as well as their weighted by M_S versions, correlated with G like M_S. The parameters $\Sigma M_i.A_I^2$ and log M_0/M_s occur useless.

5.2. Correlations for satellite systems together with planetary systems

Diagrams B11-B16 show correlations for 9 satellite systems (#21-#29, Table 3), plus Solar Systems with 4 Jovian planets only (#20, Table 2), extended by 18 planetary systems (#1-#18, Table 2). The dots (1) and circles (2) correspond to planetary systems, fitted by dashed lines. The triangles (2) correspond to systems of regular moons of solar planets plus Solar System with 4 Jovian Planets only, fitted by solid lines. The small dots (3) correspond to systems of inner small satellites of the Jovian Planets, fitted (sometimes) by short dashed lines. The numbers of the points in the diagrams correspond to the systems #20-#29 in Table 3. The open triangle in Fig. 6 and Fig. 7 (Diagrams 15 and 16), show the position of the Saturn

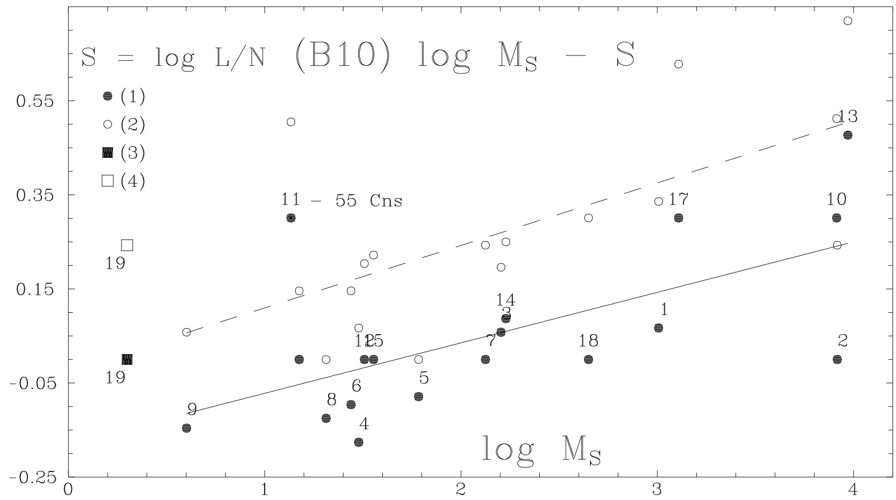

Fig. 5. Dependences of the separability parameter S (Eq.4) on the total mass of the planets, log M_S, for the main TBRs (dots, solid line) and for the alternative TBRs (circles, dashed line) for 18 planetary systems. (See Table 4.)

system according with its alternative TBR (Saturn 2, Table 3). The error bars of the gradients are shown in Fig. 6 and Fig. 7.

Diagram B11 show dependence of the total mass of the orbiting bodies on the mass of the central body. Diagram B12 shows single dependences between separability S and gradient G for the systems of planets and systems of regular moons. Dependences in the diagrams B11 and B12 are expected. However, in Diagrams 13 and 14 the correlations of S on M_0 and M_S are away.

Figure 6 (B15) shows common dependence of the TBR gradient G on the mass of the central body, log M_0. The dependence is fitted by line over 8 magnitudes of M_0. This dependence may be non-linear, but the number of the systems of regular planetary moons is small for sure conclusion.

Figure 7 (B16) shows a remarkable unique large dependence, log M_S - G, enveloping systems of planets, systems of regular moons of Jovian planets plus Pluto, as well as systems of small inner satellites of the Jovian Planets. The dependence is fitted by 3-rd order polynomial over 10 magnitudes of M_S.

In all diagrams (without B12) the ranges of the physical parameters of the orbital systems are very large. Unfortunately, in these diagrams the ranges of M_0 and M_S for the exoplanet systems are small.

6. Conclusions

In the present paper an objective method for numbering of the orbital periods and building of optimal TBRs (Sections 2, 3) is applied for 30

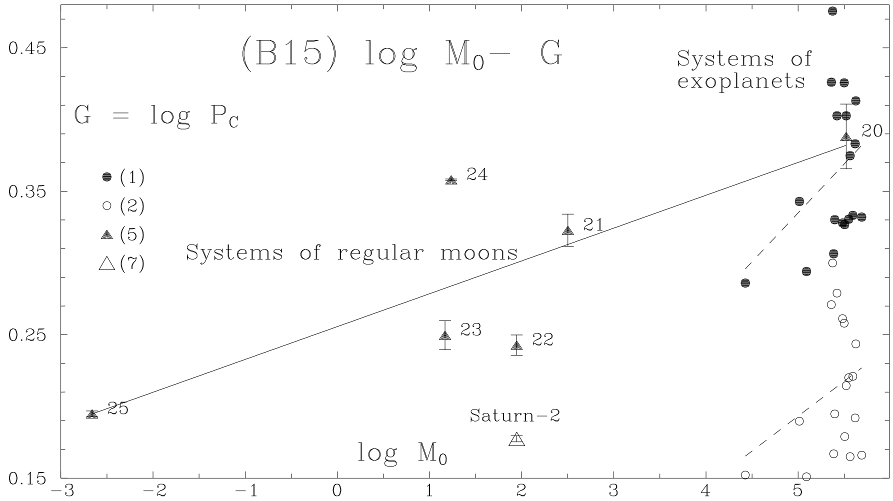

Fig. 6. Large range dependence of the TBR gradient G on the mass of the central body, log M_0. Dashed lines present linear fits for the planetary systems (dots and circles). Solid line shows the fit for the systems of the regular planetary moons (triangles).

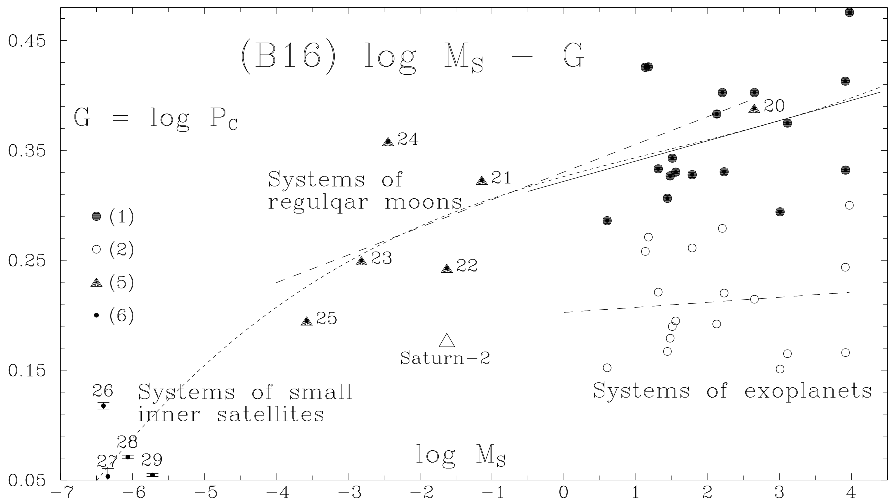

Fig. 7. Large range dependence of the the TBR gradient G on the total mass of the orbiting bodies, log M_S. Dashed lines present linear fits for the planetary systems (dots and circles). Solid line shows the fit for the systems of the regular planetary moons (triangles). Short-dashed curve presents 3-rd order polynomial fit for all orbital systems.

orbital systems (Section 4). The method is an improved version of that used earlier (Georgiev 2016). The results are as follows.

Main and alternative TBRs are revealed in all planetary systems, as well as in the complicated system of regular moons of Saturn. Only main TBRs are found for the systems of the regular moons of Jupiter, Uranus, Neptune and Pluto, as well as for the small inner satellites of the Jovian Planets (Sections 3, 4).

In the systems of solar planets and regular planetary moons the rotational period of the central body supports approximately the TBR under number $n = 0$. In contrast, in all 8 exoplanet systems with available rotational period of the star, this rotational period stands arbitrary high among the orbital periods of the exoplanets (Appendix A). While these stars are similar to the Sun, from the point of view of the TBRs the rotation of the Sun seems too slow and bounded by the orbital periods of the solar planets.

Two geometrical parameters of the TBR, gradient (Eq. 3) and separability (Eq. 4), are compared with three physical parameters of the orbital systems, mass of the central body, total mass of the orbiting bodies and (for exoplanet systems only) metallicity of the star. Positive mutual correlations in each of the 10 pairs of these 5 parameters are revealed (Table 4, Appendix B, Fig. 3, 4, 5).

The pairs of parameters separability – gradient (Fig. 4), total mass of the exoplanets – separability (Fig. 5) and metallicity of the star – total mass of the exoplanets (Diagram B3) show the best correlations for the planetary systems.

The metallicity of the star and the total mass of the orbiting bodies seem to be significant parameters for the geometry of the TBR. However, the author can not propose any explanation of this fact.

Each of the regarded 4 versions of Solar System, with 6 planets (Fig. 1), with 8 planets, with 4 Terrestrial Planets only or with 4 Jovian planets only, is similar by its TBR parameters to the exoplanet systems, regarded here.

The next task seems to be classification of the orbital systems, based on the morphology of their TBR error curves, as well as corresponding of the minima of the error curves and 3:2, 2:1, 5:2, etc., harmonic resonances of the orbital periods.

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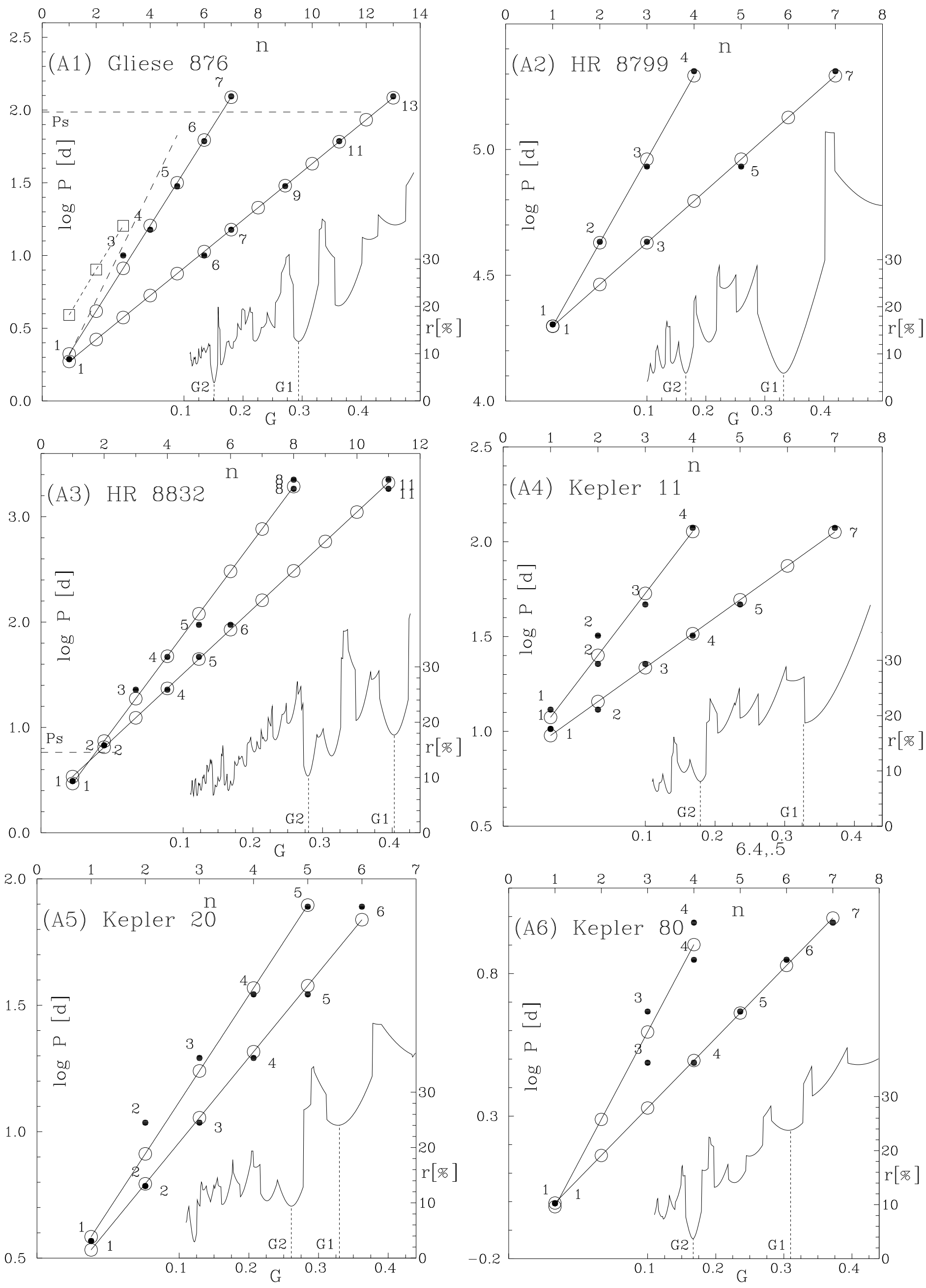
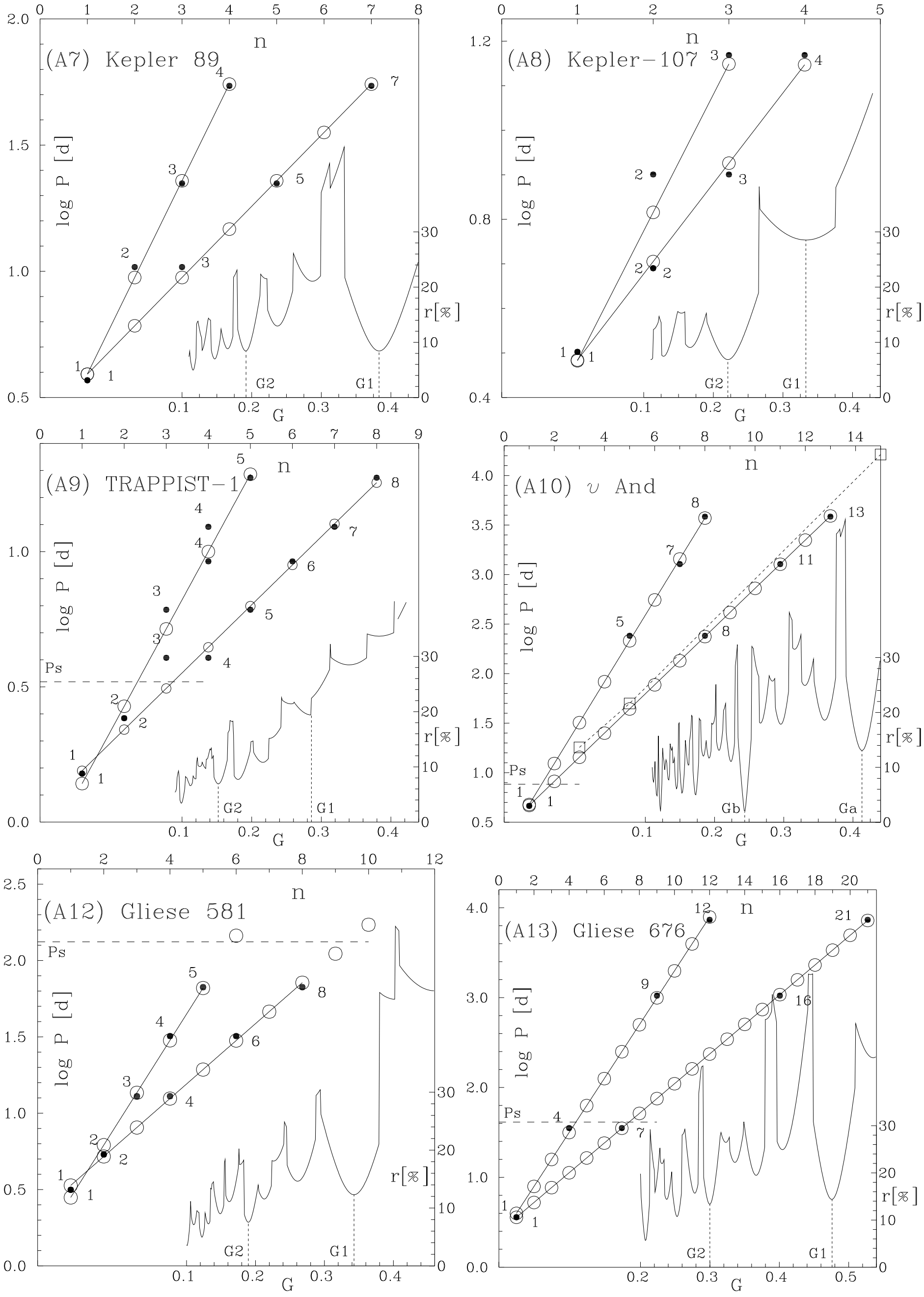
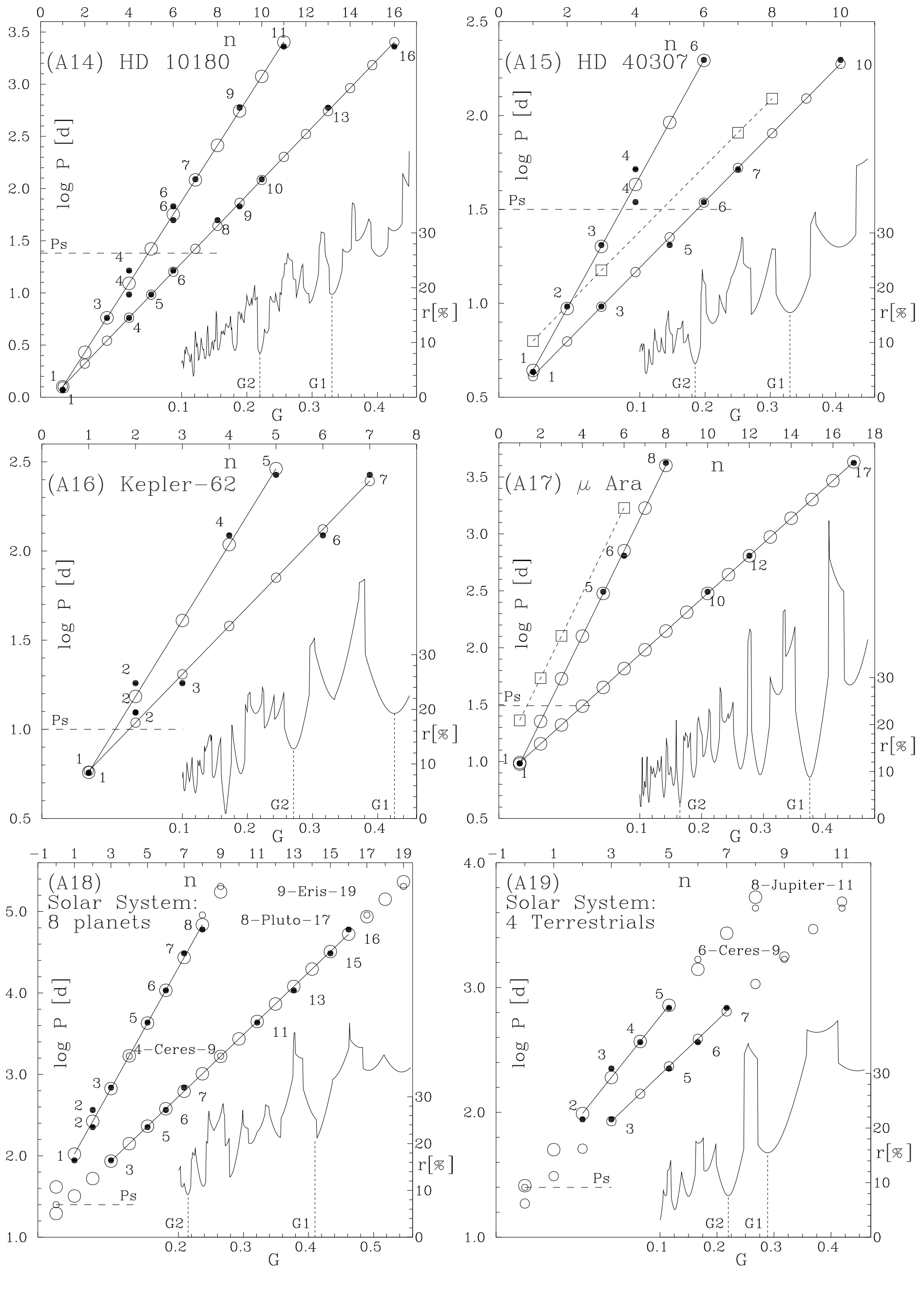
Appendix A. Main and alternative TBRs (See Section 4)



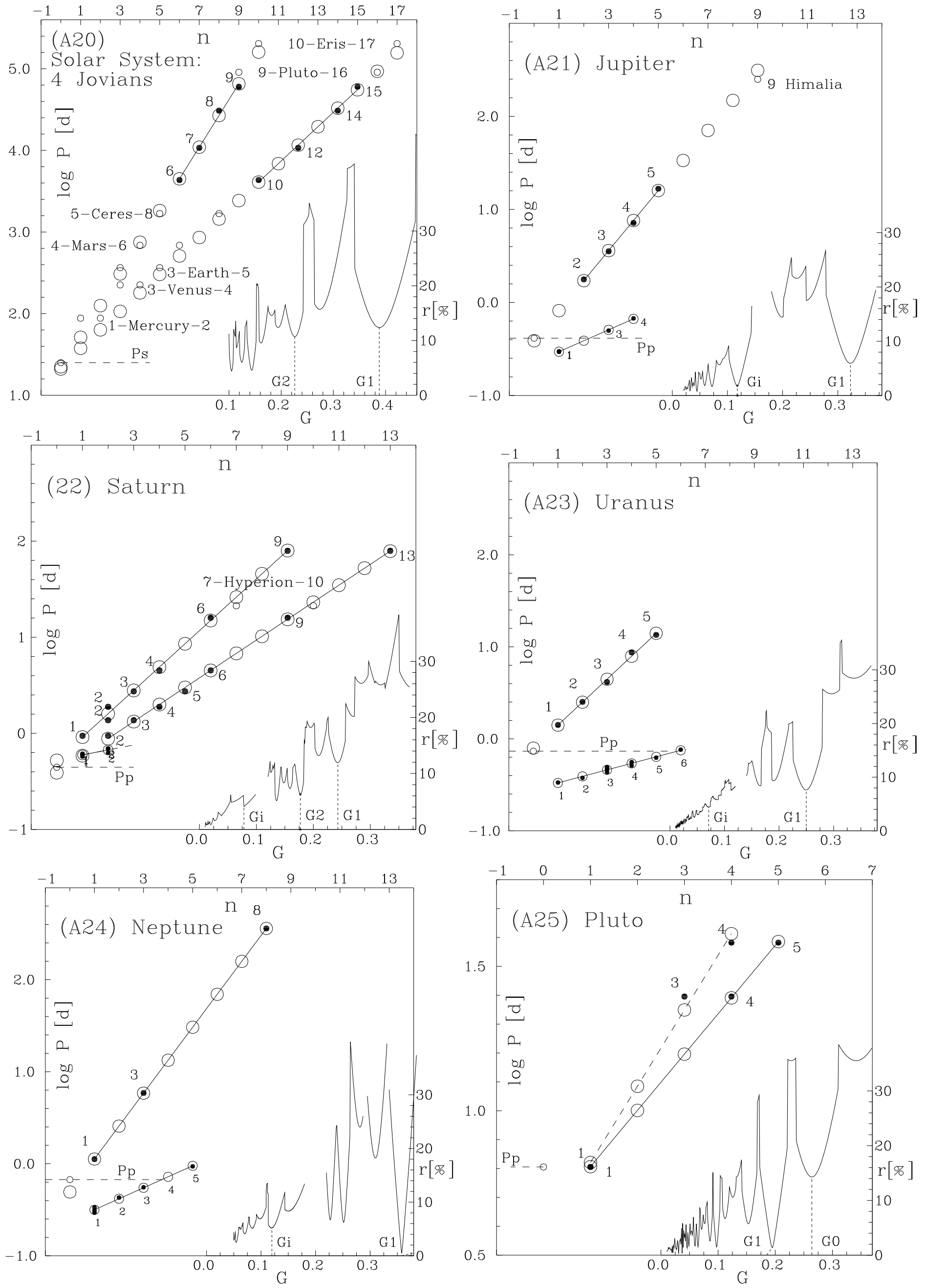
Appendix A. Continuation



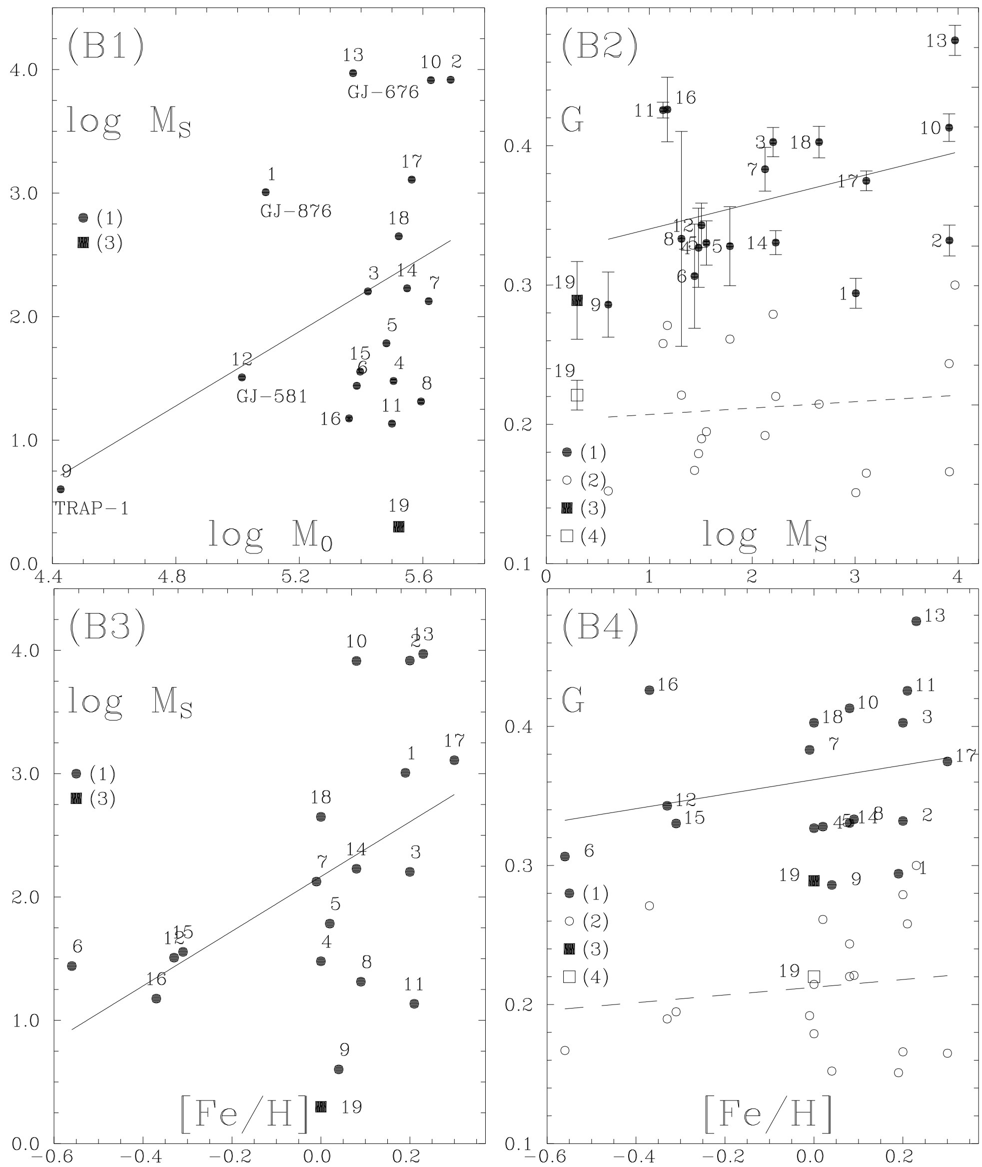
Appendix A. Continuation



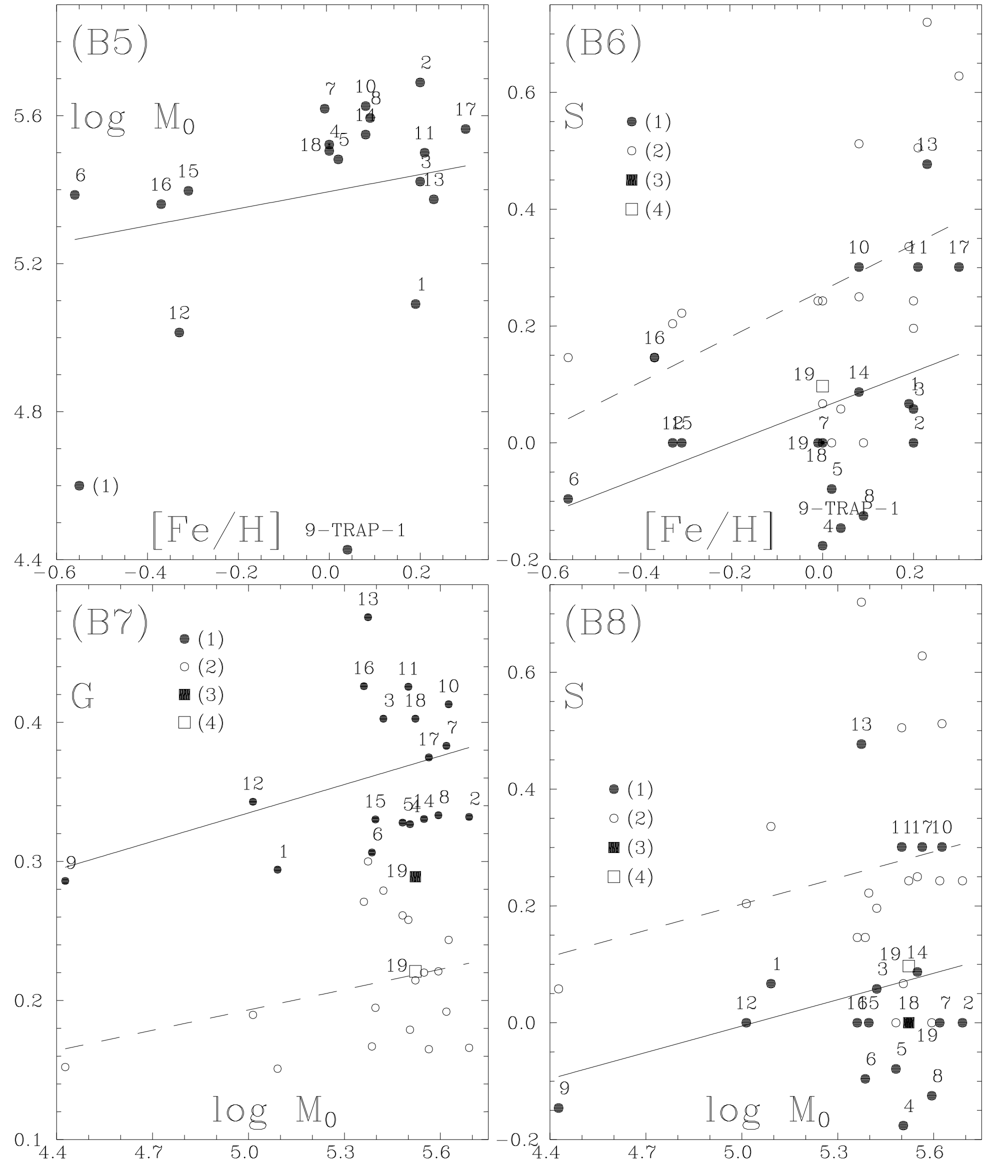
Appendix A. Continuation



Appendix B. Correlations between parameters of TBRs and parameters of orbital systems (See Section 5).



Appendix B. Continuation



Appendix B. Continuation