Applicability of QPO models to the millisecond X-ray pulsar IGR J17511-3057

Radostina Tasheva
Department of Applied Physics, Technical University of Sofia, BG-1000, Sofia
rpt@tu-sofia.bg
(Submitted on 02.08.2017. Accepted on 06.01.2018)

Abstract. Observational frequencies of the millisecond X-ray pulsar IGR J17511-3057 are used in order to test the applicability of seven differrent models of its quasy periodic oscillation (QPO). The quality of the models is tested using χ^2 - test. Our investigation confirms that there is no single model which can explain the available data i.e. they are strongly in favour of the presence of hectoHz QPOs in the spectra of the X-ray binary.
Key words: accretion disc QPOmodels X-ray binaries neutron stars

Introduction

As the X-ray timing satellite (RXTE) has revealed low mass X-ray binaries tend to display kHz QPOs in their X-ray lightcurves. Their frequencies range from several hundreds to approximately thousand Hz. The corresponding time intervals hint that we could attribute kHz QPOs to events happening near the innermost stable circular orbit (ISCO) of the accretion disk i.e. the investigation of these oscillations has the potential to help in assessement of the gravitational field of the central object (black hole (BH) or neutron star (NS)).

Cases where kHz QPOs occur in pairs may be very useful as a testing ground for the models trying to provide information about the parameters of the neutron stars or black holes. Studying them together leads to restrictions that allow the ascertaining of the mass and the spin of the central object inside the accretion disc. The case when they appear in more than one pair is a special one – the obtained results can be compared. A good match can serve as a proof for the validity of the corresponding model.

The low mass X-ray binary IGR J17511-3057 has been discovered in 2009 and the available data firstly analyzed in details by Kalamkar at al. (2011). They have observed three simultaneous pairs of kHz QPOs in X-ray spectrum of the millisecond pulsar. In our paper we use this information to test the quality of different models studied in the past by Lin et al. (2010) and Torok et al. (2012) to choose the model which is the best in explaining its QPOs. They also help us learn more about the object. If we succeed in choosing the best model it may shed some light on the viability of the models that try to explain the twin QPOs and different scenarios for the identification of QPOs proposed by Kalamkar at al. (2011). In this paper we extend the work of Stefanov (2016) by adding to the already investigated by him relativistic precession model (RP) new models, namely the modified versions of relativistic precession model, namely RP1 and RP2; the tidal disruption model (TD); the total precession model (TP) proposed by Stuchlik et al. (2007) and its version TP1; the warped disc model (WD). We also use the three pairs of data simultaneously in order to obtain a quantitative estimates of the quality of the models using χ^2 - test.

Our work is organized as follows: In section 1 we briefly present the observed twin kHz QPOs of J17511-3057 and comment on the different scenarios given by Kalamkar et al. (2011). In Section 2 the description of 7 different models for explanation of the observed QPOs is given. Section 3 is dedicated to the form in which the χ^2-test is applied to the object. Section 4 comments the applicability of the different models according to the χ^2-test and some possible reasons for the discrepancies that arise. In Section 5 we dwell on the angular momentum-mass relation (a-M) and its predictive abililty in the case of J17511-3057. In section 6 resonant switch model is applied and the results are briefly discussed. Last section is the conclusion. After the bibliography are given the equations for the fundamental frequencies of the accreted matter ν_k. ν_r and ν_θ (for more explanation see section 2). All the masses in the paper are given in solar masses and all the radii are dimensionless – $x = r/r_g$, where $r_g = GM/c^2$. The angular momentum $a = J/cM^2$ and units in which c=G=1 are used.

1. Observations

Kalamkar et al. (2011) have analyzed 71 pointed observations of the millisecond X-ray pulsar JGR J17511-3057 taken by means of RXTE PCA from September 12 to October 6 2009. The data are distributed in 7 groups which contain observations similar in time and color (for details see the original paper). In order to fit the power spectrum of each group multi-Lorentzian function is constructed and different Lorentzian is responsible for every component in the power density spectrum. We take into consideration only the components recognized as twin kHz QPOs. They appear in groups 1, 2 and 7 (see Table1). As Kalamkar et al. (2011) have mentioned the average observed power spectrum resembles, especially at low frequencies these of accreting millisecond pulsars and atoll sources in the state of extreme island state (EIS). They propose 4 possible scenarios that could explain the available components. Scenarios 1 and 2 are nearly similar – according to them the high frequency QPO seen in groups 1,2 and 7 are twin kHz QPOs. The problem here is that twin QPOs are not expected in EIS.

Table 1. Observational data containing kHz QPOs

group	$\nu_L, [Hz]$	$\nu_U, [Hz]$
1	$139.7 \pm 4,2$	$251.8 \pm 13,9$
2	$129.9 \pm 11,0$	$272.2 \pm 13,9$
7	72.5 ± 4.9	$179.9 \pm 14,9$

Scenario 3 describes the highest frequency QPOs as ν_U – the upper HF QPOs but the next to them are hecto Hz. In scenario 4 in groups 1 and 2 only hecto Hz QPOs are identified and in group 7 the two highest frequencies cannot be classified. In order to properly identify $a-M$ *relation* it is necessary to recognize the observed frequencies as twin kHz QPOs that correlate but not as hectoHz. The fourth scenario of Kalamkar at al.(2011) not only suggests availability of hectoHz QPOs in groups 1 and 2 but also excludes QPOs in group 7 because they are not harmonically related. Such hectoHz will not allow to estimate the mass and the angular momentum of the object.

2. KHz QPO models

Most hypotheses concerning QPO assume a direct connection between QPO frequencies and the fundamental frequencies of motion of the accreted matter orbiting black holes or neutron stars - the orbtital frequency ν_k.the radial frequency ν_r and the vertical frequency ν_θ . The models consider either "hot spot" or "disc" oscillation modes in QPO interpretation.

"Hot spots" are blobs orbiting at different radii x in the inner part of the accreting disc. In some cases larger homogeneities disrupted by the tidal forces (TD) of the central object can be involved in forming QPOs. According to the RP model proposed by Stella et al. (1998, 1999) periastron precession of relativistic orbits is responsible for the correlation between twin kHz QPO peak frequencies. The total precession model (TP, Stuchlik et al. 2007) and its modification TP1 assumes $\nu_\theta - \nu_r$ as total precession frequency and ν_U is ascribed to either ν_θ (TP model) or ν_k (TP1 model).

For "disc" models either axisymmetric or warped by non-axisymmetric rotation modes (WD, Kato, 2001) can be assumed. Disc resonant models suggest resonance between fundamental oscillation modes of the accretion disc orbiting the central object. The disc oscillation concept may prove to be a useful tool for explanation of the kHz QPO coherence time observed in some NS binaries (Barret et al. 2005).

RP1 and RP2 are modified versions of the RP model. In RP1 (Bursa, 2005) and RP2 (Torok et al., 2011) slow rotation case coincides with RP. The upper frequency ν_U is either ascribed to the vertical modes (RP1) or is obtained by resonance between the precession and the vertical modes (RP2). In the case of RP1 the disc torus has slight eccentricity. Table 2 includes the relations that define the lower ν_L and the upper ν_U frequency for each of the listed above models in terms of orbital frequencies. For more details about the models see Torok et al., (2011), Lin et al., (2010), Stuchlik et al., (2012).

3. χ^2 -test for experimental data retrieval

The models cited above and their combinations are used by different authors in order to explain the QPOs and to obtain the main parameters of the object – its mass and angular momentum. The reliability of their results is usually proved by other methods – for example mass can be evaluated using spectrophotometry. Our aim in the present paper is to try a mathematical approach -χ^2 -test so that we can select the best model for this accreting binary. Some kHz QPOs show 3:2 commensurability between the ν_L and the upper ν_U frequency but it is not always the case. Generally one of the

Table 2. Models and expessions for ν_L and ν_U

model	ν_L	ν_U
RP	$\nu_k - \nu_r$	ν_k
RP1	$\nu_k - \nu_r$	ν_θ
RP2	$\nu_k - \nu_r$	$2\nu_k - \nu_\theta$
TP	$\nu_\theta - \nu_r$	ν_θ
TP1	$\nu_\theta - \nu_r$	ν_k
WD	$2(\nu_k - \nu_r)$	$2\nu_k - \nu_r$
TD	ν_k	$\nu_k + \nu_r$

frequencies, for example ν_L can be expressed as a function of the value ν_U in the following way:

$$\nu_L = f(a, M, \nu_U), \tag{1}$$

Here M and a are both free parameters - the mass and the specific angular momentum of the neutron star. The χ^2 - test allows us to use the experimental values of the frequencies $\{\nu^{obs}_{L,i}, \nu^{obs}_{U,i}\}$, $i = 1, 2, 3 \ldots N$ in order to find the optimal values M_{opt} and a_{opt} for the free parameters in each model. These parameters determine the line of best fit for the relation $\nu_L = f(\nu_U)$. We can of course consider the relation $\nu_U = f(\nu_L)$ but the equations that have to be solved happen to be more complex. In order to check the viability of the models we have to minimize the function:

$$\chi^2_L(a, M) = \sum_i \left(\frac{\nu_L(a, M, \nu_U) - \nu^{obs}_{L,i},}{\sigma_{U,i}} \right)^2 \tag{2}$$

Here ν_L is the dependent variable and ν_U – the independent variable. The expression contains N=3 pairs of frequencies and M=2 free parameters (a, M) i.e. N-M=1 degree of freedom i.e. the acceptable values for χ^2 are $0 \le \chi^2 \le 2.71$.
Both variables are determined with relative uncertainty less than 10% which may still be enough to affect the value of approximation. The uncertainty of ν_U – the independent variable is neglected in the calculations.
The evaluation of ν_L is done by solving together the equations:

$$\nu_L(a, M, x) = \nu^{obs}_L \tag{3}$$

$$\nu_U(a, M, x) = \nu^{obs}_U \tag{4}$$

If we accept that both frequencies originate at the same radii we can express the radius x_U, which coincides with the radius x_L:

$$\nu_U(x) = \nu^{obs}_U \Rightarrow x_U = f(\nu^{obs}_U) \tag{5}$$

Then the calculated lower frequency ν_L depends on the observational higher frequency ν^{obs}_U through the radius x_U

$$\nu_L(\nu^{obs}_U) = f(x_U(\nu^{obs}_U)) \tag{6}$$

4. χ^2 -test results and discussion

The results from χ^2 -test for the seven chosen models – RP, RP1, RP2, TP, TP1, WD and TD are given in the Table 3. Here χ^2_{min} is the minimum value for χ^2 and a_{opt} and M_{opt} are the corresponding values for the angular momentum a and the mass M.

Table 3. Models and expressions for ν_L and ν_U

model	χ^2_{min}	a_{opt}	M_{opt}	$\chi^2, a = 0$
RP	6.88	0.83	13.89	8.44
RP1	8.15	0.32	7.20	8.44
RP2	8.05	0.12	6.76	8.44
TP	8.44	0.00	6.07	8.44
TP1	8.24	0.47	9.99	15.47
WD	7.96	0.00	5.37	7.96
TD	22.55	0.00	0.70	22.55

The last column presents the case for χ^2 when no rotation is involved i.e a=0. The calculated χ^2_{min} vary between 6.88 and 22.55 and are much greater than the appropriate boundary $\chi^2 \leq 2.71$ for a system with one degree of freedom. The lowest value for $\chi^2_{min} = 6.88$ comes from the RP model but it corresponds to the biggest optimal mass obtained– $M = 13.89M\odot$. For five of the models – RP1, RP2, TP, TP1 and WD χ^2_{min} is close to 8 and to the value for a=0, namely $\chi^2 = 8.44$. In three of the cases – RP1, RP2 and TP a_{opt} vary between 0.12 and 0.47. In five of the cases a_{opt} is smaller than 0.4. χ^2 test is, therefore, in the case of IGR J17511-3057 more in favour of slow rotation or no rotation at all. The value a_{opt}=0.83 for the RP model suggests fast rotation, but yields mass too big for a neutron star.

Optimal mass evaluations that come from different models vary between $M = 0.7M\odot$ and $M = 13.89M\odot$. The lowest mass estimate comes from TD model and according to the present theory of stellar evolution is too low for a neutron star. This value also corresponds to the highest result for $\chi^2_{min} = 22.55$, which makes this particular prediction highly improbable. The other masses are too high for a neutron star and possible reasons will be discussed briefly below.

As it is obvious the estimates for a=0 coincide for models RP, RP1, RP2 and TP because non-rotational case results in $\nu_k = \nu_\theta$.

The graphs representing the dependence of the lower QPO frequency ν_L as a function of the upper frequency ν_U according to different models are given in Fig.1 – for the RP models; in Fig.2 for the TP models; and in Fig.3 – for the WD and TD models. The presented parts of the theoretical curves are constructed according to the corresponding models and the upper frequency is limited to the observational interval - $\nu_U \in (140, 180)Hz$. The resulting lower frequency lies in the range $\nu_L\ in(60, 180)Hz$. The positions of the experimental data from groups 1, 2 and 7 are depicted along with their uncertainties.

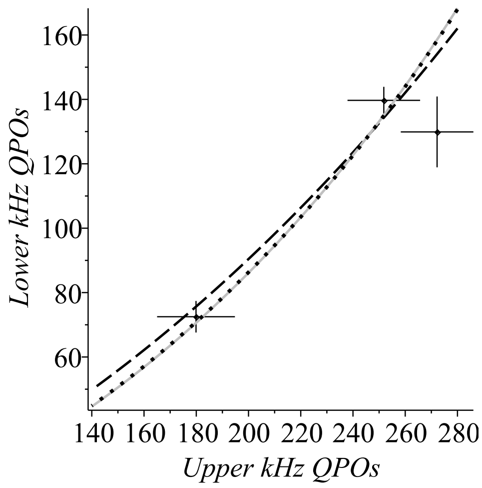

Fig. 1. Lower frequency ν_L as a function of the upper frequency ν_U according to RP (dashed black line), RP1 (solid gray line) and RP2 (dotted black line) models.

The corresponding curves for RP models are (see Fig.1) for RP – with dashed black line, for RP1 – with solid gray line and for RP2 – with dotted black line. As it is obvious from the Fig.1 theoretical predictions for RP1 and RP2 practically coincide except for the highest frequencies, which is in agreement with works of Bursa (2005, for RP1) and (Torok et al., 2011 RP2) that discuss the slow rotation case. Experimental data from groups 1 and 7 are better compatible with these models than with RP model.

In Fig. 2 the curves for the TP (solid gray line) and TP1 model (dotted black line) differ slightly in the chosen range and again group 2 seems less agreeable with theoretical predictions of these models (see Fig. 2).

Fig. 3 showns a marked discrepancy between the experimental frequencies and the WD (dash dotted black line) and TD (dashed gray line) models (see Fig.7). The TD model that produces the biggest $\chi^2_{min} = 22.55$ according to χ^2 test shows a curve that is completely detached from the experimental data, despite of the fact that it corresponds to an optimal mass $M_{opt} = 0.7M\odot$ too low for a neutron star. In the case of the WD model even if the smaller $\chi^2_{min} = 7.96$ shows better precission, the results seem to be far from accurate. The experimental frequencies of group 1 show some agreement with the predictions of the TD model, but not with those of WD model.

The conclusions that have been drawn about the pairs of simultaneous QPOs observed in the power density spectrum of the X-ray pulsar IGR J17511-3057 in the work of Stefanov (2016) show three main problems. Firstly, every combination of groups – 1 and 2, 2 and 7 and 1 and 7 give different predictions for a-M relation according to RP model. Only the latter group offers almost compatible results except for the highest values.

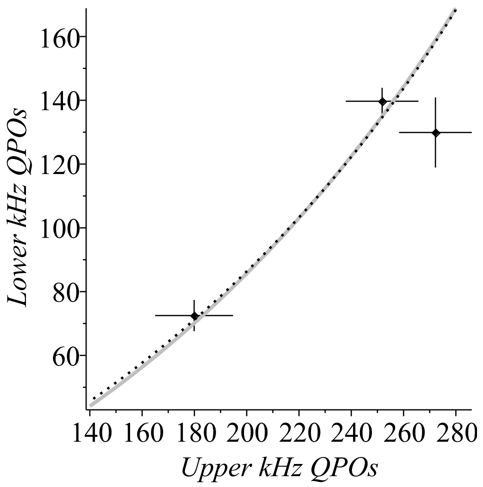

Fig. 2. Lower frequency ν_L as a function of the upper frequency ν_U according to TP, (solid gray line), TP1 (dotted black line).

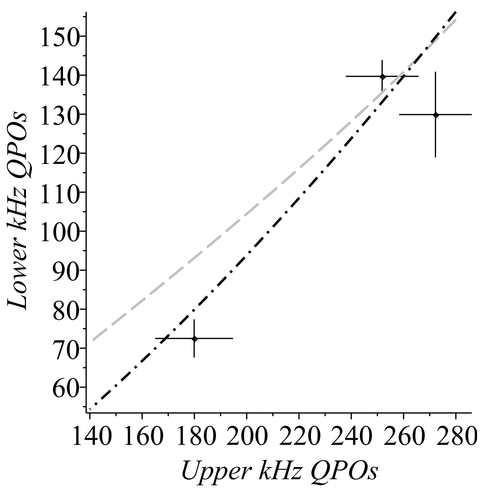

Fig. 3. Lower frequency ν_L as a function of the upper frequency ν_U according to TD (dashed gray line) and WD (dashdotted black line) models.

We obtain similar results for compatibility of observational data for groups 1 and 7 and the curves of all the models in Fig1, 2 and 3. Secondly he obtaines that the mass estimates coming from a-M relations are too high for a neutron star.

According to Stefanov, (2016) there are four main factors that can affect the obtained results – the choice of metric i.e. appropriate space-time for the region where the binary is situated, the choice of model (or models) that can explain the resulting QPOs, the uncertainty observational data are evaluated with and the choice of scenario that interprets the observed QPOs. Stefanov (2016) shows that the uncertainty of the observational data do not affect significantly the results for the mass.

As Urbanec et all. (2013) have pointed in the case of slow rotating near maximum mass NS ($a_{opt} < 0.4$ and $M \approx 3M\odot$) Kerr geometry can be applied. According to our results Kerr space-time geometry is applicable for most of the cases in our investigation. The values for the angular momentum are $a_{opt} < 0.4$ except for TP1 ($a_{opt} = 0.47$) and RP ($a_{opt} = 0.83$).

In the present paper the obtained results for the mass of the object are similarly high, for each model except for TD. χ^2_{min} are also high. Thus the probability of such masses i.e. the viability of the given models seems to be not very promising. One way out of the situation where no available model can describe the observed parameters of the neutron star may offer the resonant switch mechanism suggested by Stuchlik et al.(2012). They believe that at the resonant point occurs a switch of oscillatory modes. Due to non-linear resonant phenomena a new oscillatory mode (or modes) is created and one or more of previous modes is damped. According to Stefanov (2016) the ratio ν_U/ν_L being < 2 for group 1 and > 2 for groups 2 and 7 may be an indication that the switch point is somewhere between groups 2 and 7.

Still if we assume that only one model, but not combination of models has to explain the observed QPO frequencies, only one option remains possible – the observed QPO are not (only) kHz twin QPOs. Some hectoHz QPOs are present according to scenario 3 or all QPOs are hectoHz QPOs according to scenario 4 of Kalamkar et al. (2011). If former or latter scenario is applicable is yet to be determined, but discrepancies between the prediction coming from groups 1 and 2 on one hand and 7 on another (Kalamkar el al. 2011) suggest that scenarios 3 or 4 are more possible. The work of Stefanov (2016) as well as the results of our paper are in favour of these scenarios.

5. Mass prediction according to a-M relation

If we solve the system (3)-(4) having fixed the radius x=r/rg a pair of solutions for a and M will be produced. If x is also varied a family of solutions will be obtained defining a curve in the a-M space. Having in mind that available data from the observational frequencies are determined along with their uncertainties (see Table 1), a strip is to be formed between the two curves, with width depending on the on the chosen values of the observational frequencies.

Group 1 observations obviously give the lowest mass prediction for all the models, which is always bigger than $4M\odot$. WD and TD models make exception. The WD model (see Fig. 4) yields mass estimate somewhere between $3M\odot$ and $4M\odot$ when $a < 0.5$ and the prediction for the lowest mass comes from the group 2 observations. The tidal disruption model (see Fig. 5) gives masses between $0.3M\odot$ and $4.5M\odot$ when $a < 0.5$ but only for group 1 observations. There are no positive mass solutions for other

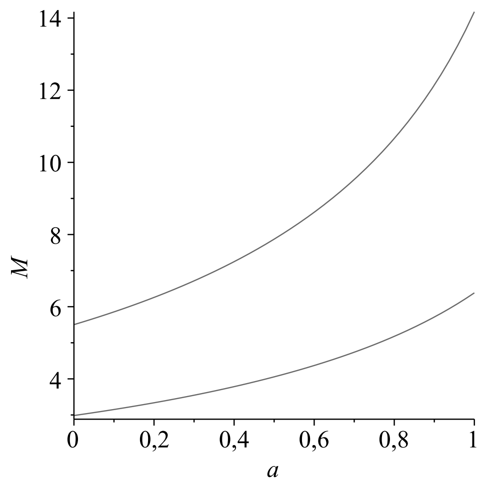

Fig. 4. a-M realtion to for the case of WD model. The model predictis least possible mass $M \approx 3M\odot$

two groups. This result is consistent with the big $\chi^2_{min} = 22.55$ and the deviation of group 2 and 7 data from the experimental curve for the TD model (see Fig. 3).

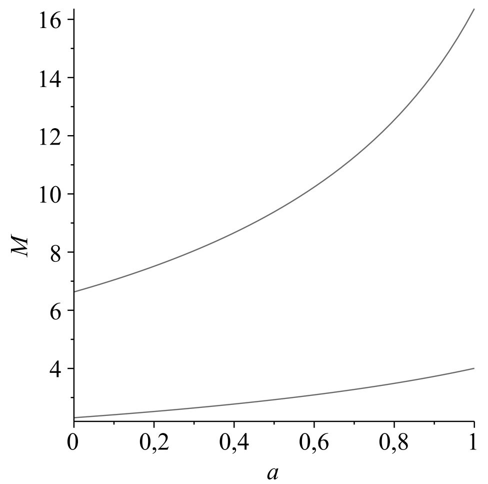

Fig. 5. a-M realtion to for the case of TD model. The model predictis least possible mass $M \approx 0.3M\odot$

6.Resonant switch (RS) model for IGR J17511-3057

Application of RS model is done according to Stuchlik at al. (2013). The outer resonant point has frequency ratio $\nu_U/\nu_L = 5 : 2$ and the inner resonant point - frequency ratio approximately $\nu_U/\nu_L = 2 : 1$. The switch resonant point can be determined by the place where these curves are crossing according to experimental data. For RP-TP switch model TP is related to the inner resonant point and RP to the outer, similar is the case for RP1-TP and RP1-TP1 switch models. The RP-TP combination do not give decisive results for the switch resonant point. The range of mimimum masses sugested by the other two models vary as $5M\odot < M < 9M\odot$. The possible diapason for the angular momentum is $0.1 < a < 0.8$ for the former models and $0 < a < 0.6$ for the latter (See Fig. 6 and 7). The central point - the point where the central lines of the both stripes cross suggests approximately $M = 6M\odot$ and $a = 0.5$ for RP1-TP model and $M = 5.7M\odot$ and $a = 0.3$ for RP1-TP1 model.

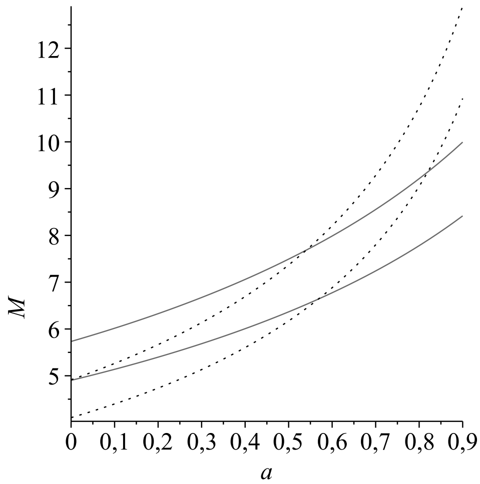

Fig. 6. a-M realtion to for RP1- TP switch model. The model predictis mass $M \approx 6M\odot$ and $a \approx 0.5$

Obviosly the so obtained masses are again bigger that usual $M = 3M\odot$ boundary and combinations between RP and TP models and their modifications are more likely to be excluded as possible explanations of the obtained frequency data. RP-TD combination should be also excluded due to lack of positive mass solutions for the TD model. If the data for the low mass binary IGR J17511-3057 from the work of Kalamkar (2011) are to be compared with the data for the Circinus X1 source where also simultaneous twin oscillations are observed one may expect that similar QPO frequencies should lead to similar mass estimates – about $M = 6M\odot$. There are though very important differences in observational data for these objects that could

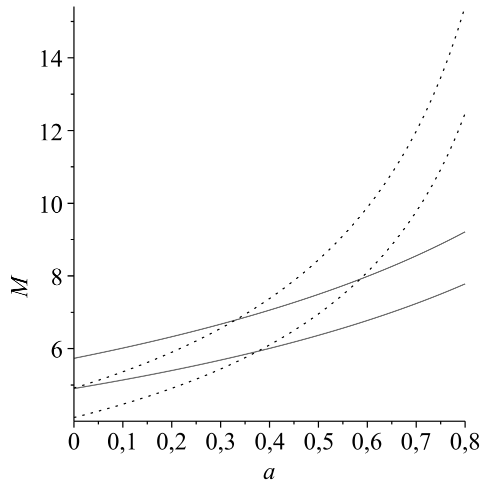

Fig. 7. a-M realtion to for RP1- TP1 switch model. The model predictis mass $M \approx 5.7 M\odot$ and $a \approx 0.3$

lead to differences in their parameters. While the frequencies of groups 1, 2 and 7 for IGR J17511-3057 are somewhere around $\nu_U/\nu_L = 2:1$ ratio, those for Circinus X1 source are between $\nu_U/\nu_L = 3:1$ and $\nu_U/\nu_L = 4:1$ which may significantly alter the masses and angular momenta obtained i.e in our case to increase the mass estimates.

Conclusion

This paper is one extension of the work of Stefanov (2016) in the direction of QPO model application. We apply seven models, namely RP, RP1, RP2, TP, TP1, WD and TD models to the assumed twin kHz QPOs that occur simultaneously in groups 1, 2 and 7 of the observations of power density spectra of the X-ray millisecond pulsar JGR J17511-3057. Using χ^2 - test we try to find the best model capable of explaining the observational data. All the models according to the χ^2 - test fail to fit the ensemble behavior of the lower ν_L and the upper frequency ν_U QPO frequencies. The minimal values χ^2_{min} are much bigger than appropriate for a system of variables with one degree of freedom $\chi^2_{min} \leq 2.71$. The obtained optimal angular momenta a_{opt} vary between 0 and 0.83, but in five of the cases is smaller than 0.4 suggesting slow rotation or non-rotational case. The optimal masses M_{opt} are much higher than usual for a neutron star ($M < 3M\odot$) except for the case of TD model where $M_{opt} = 0.70 M\odot$ but accompanied by the biggest $\chi^2_{min} = 22.55$.

The a-M relations also predict bigger than $M = 3M\odot$ in the cases of RP and TP model and their modifications. Even if the predicted mass is near $M = 3M\odot$ – bigger for WD model and smaller for TD model the viability of this models is yet to be determined. An application of the RS models according to Stuchlik at al. (2013) attempted in order to find a hybrid model capable of solving the mass problem also have failed giving results impobable for a neutron star.

Some of the possible reasons for the failure as the improper choice of metric and the role of the uncertainties of the observational data are already discussed in details in the cited above works. Kalamkar et al.(2011) suggest scenarios according to which the observed QPOs may not be twin kHz but hectoHz QPOs that are not harmonically related. Our results show that according to χ^2 - test there is no single model which can explain the available data i.e. they are strongly in favour of the proposal of Kalamkar et al.(2011) about the presence of hectoHz QPOs in the spectra of the X-ray millisecond pulsar JGR J17511-3057.

Acknoledgments

The author would like to thank Ivan Stefanov for his invaluable discussion on the manuscript. We also thank the referee, prof. Stuchlik who provided constructive feedback that helped to significantly improve the paper. This research is partially supported by the the Bulgarian National Science Fund under Grant No N 12/11 from 20 December 2017.

References

Barret D., Kluzniak W., Olive J., Paltani S., Skinner G., 2005, *MNRAS, 357, N4,1288*
Bursa, M. 2005,*In Proceedings of RAGtime,* 6/7 *, Opava, editors S. Hledik and Z. Stuchlik*
Kalamkar M., Altimirano D., van den Klis M., 2011, *ApJ 729, 7*
Kato S., 2001, *ApJ, PASJ,53,1*
Lin Y, Boutelier M., Barret D., Zhang S., 2010, *arXiv:1010.6198v1astro-ph.HE*
Stella L., Vietri M., 1998, *ApJ, 492, L59*
Stella L., Vietri M., Morsink S., 1999, *ApJ, 534, L63*
Stefanov I, 2016, *Astron.Nachr./AN 337, N3, 246*
Stuchlik Z., Torok G., Bakala P., 2007, *arXiv:0704.2318*
Stuchlik Z., Kotrlova A., Torok G., 2012, *Acta Astron, 62, 389*
Stuchlik Z., Kotrlova A., Torok G., 2013, *A&A, 2013, 552, A10*
Torok G., Kotrlova A., Sramkova E., Stuchlik Z., 2011, *A&A , 531, A59, arXiv:1103.2438.*
Torok G., Bakala P., Sramkova E., Stuchlik Z., Urbanec M., Goluchova E., 2012, *A&A, 531, A59, arXiv:1103.2438.*
Urbanec M., Miller, J. C., Stuchlık Z., 2013, *MNRAS, 433, 1903*

Appendix 1. Fundamental frequencies

$$\nu_k = \frac{1}{2\pi} \times \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}}$$

$$\nu_r^2 = \nu_k^2 (1 - \frac{6M}{r} - \frac{3a^2}{r^2} \pm \frac{8aM^{1/2}}{r^{3/2}})$$

$$\nu_\theta^2 = \nu_k^2 (1 + \frac{3a^2}{r^2} \mp \frac{4aM^{1/2}}{r^{3/2}})$$