Influence of celestial parameters on Mercury's perihelion shift

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Dedicated to Jan Brandts on his 50th birthday

Abstract. This paper considers the influence of numerical values of the celestial parameters on the indeterminacy of the perihelion shift of Mercury's orbit. This shift is thought to be one of the fundamental tests of the validity of the general theory of relativity. In the current (astro)physical community, it is generally accepted that the additional relativistic perihelion shift of Mercury is the difference between its observed perihelion shift and the one predicted by Newtonian mechanics, and that this difference equals $43''$ per century. However, as it results from the subtraction of two quite inexact numbers of almost equal magnitude, it is subject to cancellation errors. As such, the above accepted value is highly uncertain and may not correspond to reality. We present a thorough numerical analysis of this problem.
Key words: Solar system, ephemeris, planets, line of apsides, perihelion advance, heliocentric system

1. A brief historical overview

The French astronomer Urbain Le Verrier (1811–1877) is best known for his prediction of the position of Neptune (at that time an unknown planet). He performed his calculations to explain certain irregularities in the orbit of Uranus. In 1859, Le Verrier also noticed some anomalies in the observed position of Mercury's perihelion compared to Newton's theory (Le Verrier, 1859). According to (Tisserand, 1880, p. 36), he needed to explain a shift of $38''$ per century to make the position of Mercury's perihelion predicted using Newtonian mechanics in agreement with the actual observations. In 1895, Simon Newcomb [1895, Chapt. IX, p. 184] arrived at the comparably very small value of

$$P = 43.37'' \text{ per century.} \tag{1}$$

In 1915, Albert Einstein published in [1915, p. 839] a formula for the relativistic perihelion shift, for one period, of

$$\varepsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1 - e^2)} = 5.012 \cdot 10^{-7} \text{ rad}, \tag{2}$$

where according to contemporary data $T = 7.6005 \cdot 10^6$ s is the orbital period of Mercury, $e = 0.2056$ the eccentricity of its elliptical orbit, $a = 57.909 \cdot 10^9$ m the length of its corresponding semimajor axis, and $c = 299\,792\,458$ m/s is the speed of light in vacuum. Substituting these values into (2), we obtain a value of

$$E = \varepsilon \frac{\tau}{T} \frac{180}{\pi} 3600'' = 43'' \quad \text{per century,} \tag{3}$$

that is in excellent agreement with the calculations of both Le Verrier and Newcomb (1). Here $\tau = 3\,155\,814\,954$ s is the number of seconds in one century.

We should stress here that over the past 100 years Mercury's perihelion was shifted by a total amount of about $5600''$ (see (Clemence, 1947, p. 363)) of which approximately $5027''$ is due to the precession of the Earth's axis. The vernal equinox (see Fig. 1), from which the equatorial right ascension is measured, moves along the ecliptic by, on average, $50.27''$ per year. This value is also continually distorted by the nutation of the Earth's axis of $\pm 9.21''$ with a period of 18.6 years. The accurate determination of the position of the vernal equinox is therefore a very difficult task. Moreover, due to non-zero inclinations of planets (Jupiter 1.3°, Saturn 2.5°, etc.), the position of the ecliptic in space changes with time. Thus, the use of a time-dependent equatorial coordinate system with right ascension and declination is not appropriate.

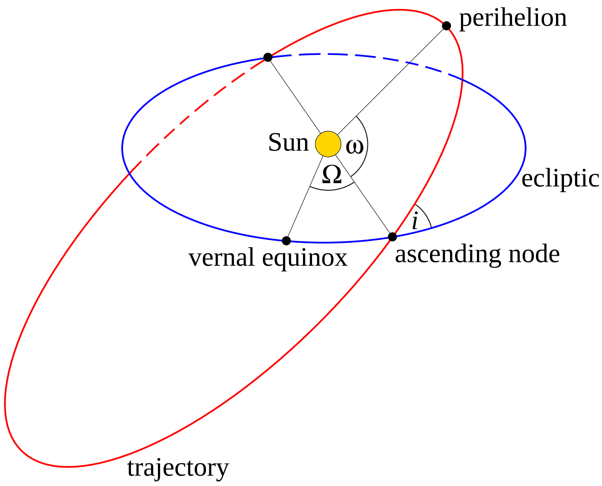

Fig. 1. Keplerian parameters of the elliptical orbit of the planet that determine its orientation in space are: the inclination i, the longitude of the ascending node Ω, and the argument of perihelion ω. They vary slightly in time due to the presence of other planets, precession and nutation of the Earth's axis.

Therefore, we will next consider a rectangular heliocentric system whose position is unchanged with respect to fixed stars. In this system, the currently observed perihelion shift of Mercury due to the gravitational pull of the other planets is about $575''$ per century which is more than a ten times larger value than that in (3). On the other hand, according to the calculations of Le Verrier [1859, p. 99], Mercury's perihelion shift is influenced by the other planets as follows:

planet	its impact on Mercury's perihelion advance
Venus	280.6″
Earth	83.6″
Mars	2.6″
Jupiter	152.6″
Saturn	7.2″
Uranus	0.1″
Total	526.7″

Tab. 1. The effect of particular planets on Mercury's perihelion shift per century by (Le Verrier, 1859)

The total sum in the last line of Table 1 is to be understood only approximately, because the problem in question is nonlinear and such a sum is not justified. Additionally, although the other values in Table 1 are given in up to four significant digits, not all of them are accurate. For instance, at the time of Le Verrier, the masses of the planets were not precisely known: Mercury was believed to have double the mass assigned to it nowadays, using the accurately measured orbital period of the Messenger satellite (Srinivasan, 2007) and Kepler's Third Law. On the other hand, the mass of the Earth was underestimated as 0.937 of its currently accepted value (Le Verrier, 1859, p. 19). Therefore, Le Verrier was not able to derive reliable values for the gravitational forces between the planets. Neither did he solve the system of ordinary differential equations describing the N-body problem, but made approximations using certain sums of finite series instead by means of his perturbation theory. Finally, he only had relevant data concerning the positions of the planets over the preceding fifty years.

The values in Table 1 are also affected by other sources of errors. For instance, the relative position of Jupiter in its orbit at the beginning of each century varies due to the simple fact that its orbital period of 11.861 years does not divide 100 years. As the same holds true for the other planets, they all influence Mercury in an irregular manner from varying positions (cf. the right part of Fig. 2). Thus, the perihelion shift can essentially differ century by century.

Our further aim will be to provide a more critical insight into the problem of Mercury's perihelion shift. In Section 2 we recall that the difference between two almost equally large numbers can be quite inaccurate. In Sections 3 and 4 we illustrate why the observed and calculated values of Mercury's perihelion shift are highly imprecise. Finally, in Section 5 we show that the simple Einstein's formula (2) was derived from the 10 nonlinear Einstein's equations by means of many approximations.

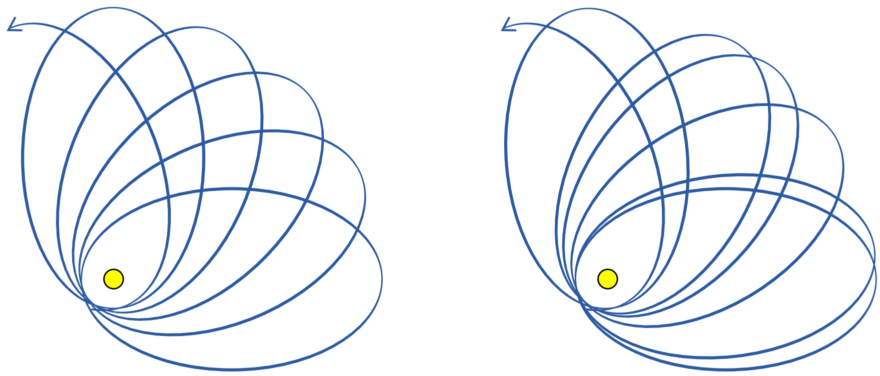

Fig. 2. On the left the idealized uniform perihelion shift of Mercury in the direction of circulation is shown. For clarity, a very high artificial eccentricity $e = 0.8$ is chosen. On the right an irregular perihelion shift caused by the gravitational tug of other planets is schematically depicted.

2. The inaccurate difference between two almost equally large numbers

It is true that the manual calculations of Le Verrier and Newcomb (cf. (1)) led to values close to (3). However, they did not reach the accuracy of simple computer arithmetic in which each number is stored in six bytes, they originated from inexact data, and were obviously subject to various errors (see e.g. (Inoue, 1993)). Therefore, we restrict ourselves to the current status quo.

Denote by the letter O the observed value of Mercury's perihelion shift per century, and by C the calculated value using Newtonian mechanics. At present, it is generally accepted that the following equality is true

$$O - C = E, \tag{4}$$

where E is the value (3) predicted by Einstein. For instance, (Narlikar, Rana, 1985, p. 657) claim that

$$O = 575'' \text{ per century} \quad \text{and} \quad C = 532'' \text{ per century} \tag{5}$$

which yields the value (3) by (4). For simplicity, most of perihelion shifts will be from now on rounded to integers given in arc seconds. We shall not present the corresponding error bars, since they are always less than $1''$ in recent literature. It is clear that if at least one term in relation (4) is not correctly established, then the proclaimed equality (4) does not bring anything useful and the value E may differ from reality.

Numerical analysts know very well (see (Brandts et. al., 2016), (Goldberg, 1991)) that subtracting two almost equally large numbers (see (4)) in floating point computer arithmetic is burdened by a large resulting error. Although generally no rounding appears when subtracting such numbers, the mantissa of the difference contains only a few significant digits which inevitably leads to loss of precision. Let us present an illustrative example.

Example. Assume for simplicity that the mantissa has only 5 digits and calculate the difference $3.1416 \cdot 10^0 - 3.1415 \cdot 10^0$. The computer stores the result as the number $1.0000 \cdot 10^{-4}$, where four zeros in the mantissa are not significant digits. This means that the error of the difference has been moved to the second digit in the mantissa (generally from the last digit of the given number).

The following three sections show that all three values in (4) are loaded with plenty of different errors, and thus the proposed equality (4) is not very likely to be numerically verifiable. ¿From the previous section we know that the observed (and also calculated) perihelion shift must differ century by century, cf. the right part of Fig. 2 and Fig. 6 below.

Furthermore, note that the full angle has over a million arc seconds, namely,

$$u = 360 \cdot 3600'' = 1\,296\,000'',$$

while equation (3) gives less than one arc second per year. Mercury's perihelion separation from the Sun is $r_1 = a - ae = 46 \cdot 10^6$ km. According to (3), the additional perihelion shift is[1]

$$\frac{2\pi r_1}{u} \cdot 0.43'' = 96 \text{ km per year.} \tag{6}$$

For comparison, the orbital speed of Mercury is about 50 km/s.

3. The observed perihelion shift of Mercury

To determine the exact position of Mercury's actual perihelion is one of the most difficult tasks in contemporary positional astronomy and computational mathematics due to a multitude of reasons. In the case of a small eccentricity $0 \le e \ll 1$, we get an ill-conditioned problem, since for the circular path the perihelion is at each point. The eccentricity of the Mercury's orbit is relatively large $e = 0.2056$, but since the semiminor axis of Mercury has the length

$$b = a\sqrt{1 - e^2} = 0.98a,$$

the orbit is almost circular with the Sun being at one of the foci.

Mercury can be seen only a few days per year in the projection on a flaring celestial sphere. Hence, it was difficult to reliably determine from Earth its angular distances from neighboring stars. Since Mercury is mostly close to the horizon at sunrise or sunset, serious difficulties were also caused by the astronomical refraction of the atmosphere:

[1] In (Misner et al., 1997, p. 1048), the additional perihelion shift (3) is inexactly established as 120 km per year.

zenith distance	mean refraction
$0°$	$0'$
$20°$	$0'22''$
$40°$	$0'50''$
$60°$	$1'43''$
$70°$	$2'43''$
$80°$	$5'30''$
$90°$	$35'$

Tab. 2. Influence of the zenith distance on the mean atmospheric refraction

Moreover, the refraction depends on the air temperature, pressure, humidity, etc. Note that the value in the last row of Table 2 is given in arc minutes, and not in arc seconds (cf. (3)). It is even larger than the angular diameter $30'$ of the Sun. Consequently, when the Sun touches the horizon at sunset and we would immediately remove the Earth's atmosphere, the Sun would already be below the horizon and darkness would occur. Fortunately, at present the precise position of Mercury on the celestial sphere is determined by satellite measurements.

To determine the exact instantaneous position of Mercury in heliocentric coordinates, we need to know precisely both the time-dependent right ascension and declination, but also the distance from the Earth, which currently can be obtained by using radar reflections (Anderson et al., 1996), (Jurgens et al., 1998), (Standish et al., 2005). However, these measurements are done in a relatively short time interval compared to one century. Moreover, we have to take into account that even the Earth moves during the measurements along a complicated nonelliptic orbit.

The observed positions of Mercury in finitely many points have to be interpolated in a rather complicated way and then converted to the heliocentric coordinate system, the center of which is the Sun and which is fixed against distant quasars. Estimation of the acceleration of the Solar-system barycenter relative to a system of reference quasars is established in (Titov, 2011).

But here arises another serious problem, because the true orbit of Mercury is not elliptical. For illustration, consider first a simple Sun-Jupiter system, where masses of the Sun and Jupiter are

$$M = 1.989 \cdot 10^{30}\text{kg}, \quad m = 1.899 \cdot 10^{27}\text{kg},$$

respectively. If we place the barycenter of this system into the origin of Cartesian coordinates, then from the equation $MR = mr$ we find that the distance of Jupiter and Sun from the origin is $r = 778 \cdot 10^6$ km and $R = 743\,000$ km, respectively, while the radius of the Sun is $696\,000$ km, cf. (6). Thus we observe that the barycenter of the Sun-Jupiter system lies outside the Sun, and the Sun and Jupiter orbit about it.

However, the Sun's trajectory is also deflected by other planets (especially those very distant), i.e., the heliocentric system is not inertial. Fig. 3 shows the projection of the trajectory of the Newtonian barycenter of the Solar system in the heliocentric system over the duration of half a century, starting from 2000. Thus, the actual trajectory of Mercury is not an ellipse, one of whose foci is placed at the Sun, but it is a very complicated nonplanar trajectory, which is influenced by the barycenter of the Solar system. It is therefore unclear how to define precisely e.g. the semimajor axis of Mercury, which according to (2) should be shifted. It is also unclear how to determine an analogy of the Newtonian barycenter of the Solar system for a finite speed of propagation of gravity.

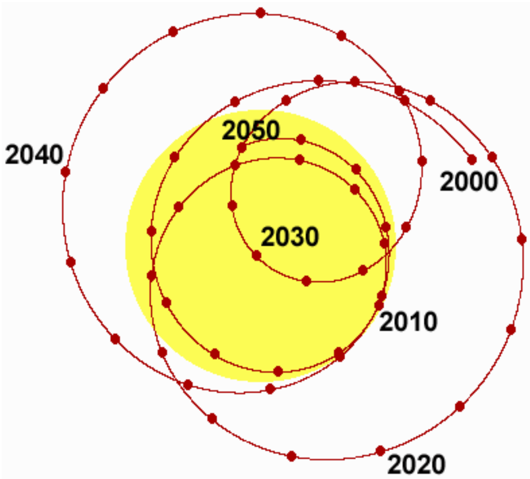

Fig. 3. Projection of the trajectory of the Newtonian barycenter of the Solar system into the ecliptic for the period 2000–2050. The center of the Sun (whose diameter is almost 1.4 million km) is at the origin of the heliocentric system. The barycenter shifts each day by about 1000 km, while the additional relativistic perihelion shift of Mercury is on average of only 96 km in a year due to (6).

Let us present other facts that need to be taken into consideration for determining the actual position of Mercury. The average speed of Mercury is about 50 km/s. Mercury's diameter is nearly 5000 km and its angular diameter when crossing the solar disk is $12''$. Before information about its position reaches our Earth by the speed of light, Mercury will move more than three of its diameters, i.e. approx $40''$ and 17 000 km (cf. (3) and (6)). But astronomical tables indicate planets where they are observed and not where they really are.

To determine the actual position of Mercury it is necessary to take into account also the aberration of light, which for the Earth's orbit is $\approx 20''$. Terrestrial measurements are also slightly affected by the atmospheric dispersion of light. The length of the visible spectrum is equal to approximately $1''$ for the angle $30°$ above the horizon. The Earth-Moon system gravitationally acts slightly unbalanced on Mercury (Clemence, 1947, p. 363). Furthermore, from precise observations we know that the measured secular acceleration of Mercury is proportional to its mean motion. Secular long term deviations in the mean longitudes of this planet have reached $5''$ during the last 250 years (Kolesnik, Masreliez, 2004, p. 884), i.e., after approximately 1000 periods. From this Igor N. Taganov [2016, p. 72] derived that Mercury's orbit expands at a the rate comparable to the Hubble constant.

We should also take into consideration the calibration of instruments, which is essential when determining equatorial coordinates of Mercury and its distance from Earth. Determining the actual orbit of Mercury from astronomical observations is therefore an extremely difficult task which has potential for different errors that have a significant impact on the value O from (4).

4. Computed perihelion shift of Mercury

Fig. 4 shows a general scheme of computational mathematics to solve real-life (non-academic) problems of mathematical physics. We always produce three types of error: the modeling error e_0, the discretization error e_1, and rounding errors e_2.

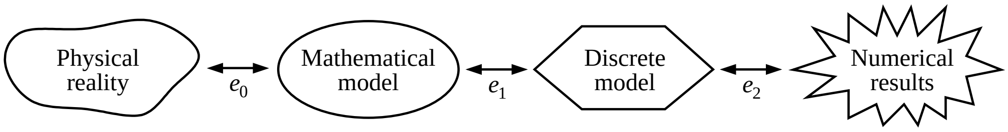

Fig. 4. The modeling error $e_0(t)$ is the difference between physical reality and its mathematical description. A discrete finite dimensional model differs from the mathematical model by the discretization error $e_1(t)$. Finally, in $e_2(t)$ are included rounding errors, iteration errors, etc.

In our case, the physical reality is the Solar system. Its evolution is usually modeled by the problem of N bodies that interact with each other gravitationally. Bodies are replaced only by idealized mass points m_i, $i = 1, \dots, N$, whose positions r_i satisfy the well-known system of second order differential equations

$$\ddot{r}_i = G \sum_{j \neq i}^{N} \frac{m_j (r_j - r_i)}{|r_j - r_i|^3} \tag{7}$$

for $i = 1, \dots, N$ with given initial conditions on the positions $r_i(0)$ and velocities $\dot{r}_i(0)$ of all N bodies. Here $|\cdot|$ stands again for the length of a vector and G is the gravitational constant.

The system (7) represents the considered mathematical model, creating a nonzero modeling error e_0. The system of differential equations is nonlinear and has many unrealistic solutions which allow highly superluminal speeds (e.g. by (Saari, Xia, 1995) five bodies can get to ∞ in finite time). Hence, it is difficult to derive a guaranteed modeling error estimate that cannot be improved. The modeling error e_0 is often ignored, since on short time intervals the Newtonian mechanics in the Solar system yields a very good approximation of reality. The modeling error e_0 is also affected by the assumed infinite speed of propagation of gravitational interaction, which violates causality. This speed is certainly finite, see (Kopeikin, Fomalont, 2006). Two LIGO detectors whose distance is $d = 3002$ km recorded in 2015 a passage of a gravitational wave. The time delay between these two detections was $\varDelta t = 6.9$ ms (see (Abbott et al., 2016, p. 2)). This immediately leads to the following upper bound on the speed of gravitational waves

$$c_{\mathrm{g}} = \frac{d \sin \alpha}{\varDelta t} \leq \frac{d}{\varDelta t} = 435\,072 \text{ km/s}$$

in vacuum or in the Earth body, see Fig. 5.

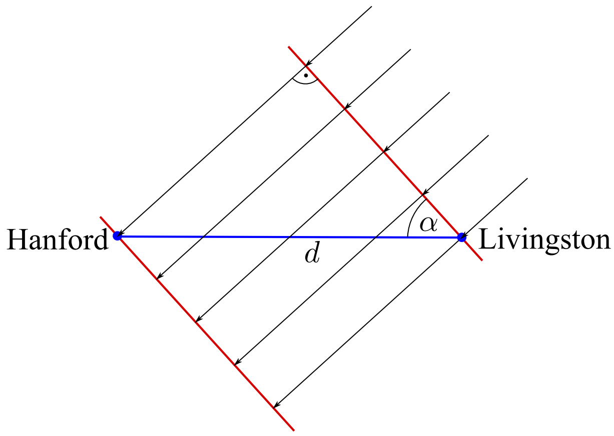

Fig. 5. Depiction of the passage of a planar gravitational wave through two detectors LIGO in USA

For $N > 2$, the N-body problem generally does not have an analytic solution and therefore, the solution is usually approximated by numerical methods. This leads to the so-called discrete finite dimensional model, which produces another nonzero error e_1. The N-body problem is not stable with respect to changes in initial conditions and continually acting perturbations. Thus each numerical integration method gives a considerable discretization error over long intervals, even if we use a double (8 bytes) or extended (10 bytes) computer arithmetic.

Finally, the error e_2 depends among other things on the used computer, programming language, and also on the fashion of programming. For a catastrophic behavior of rounding errors see (Brandts et al., 2016), (Křížek, 2015). Numerical errors usually do not average out, but accumulate and grow exponentially in the course of calculation. Each programmer (and also observer) can get different results. For instance, in Russia completely different programs are used to calculate the ephemerides of planets (Pitjeva, 2005) and (Pitjeva, Pitjev, 2014) than in France (Bretagnon, Francou, 1988) or in the NASA Jet Propulsion Laboratory, Pasadena, USA, see e.g. (Standish, 2004) and (Standish et al., 2005).

The difference between the exact and approximate solution of a mathematical model is estimated in numerical mathematics by a priori or a posteriori error estimates. This step is very important. If we do not perform a reliable analysis of errors, we do not know actually what we have calculated, and how far the numerical solution is from the exact solution.

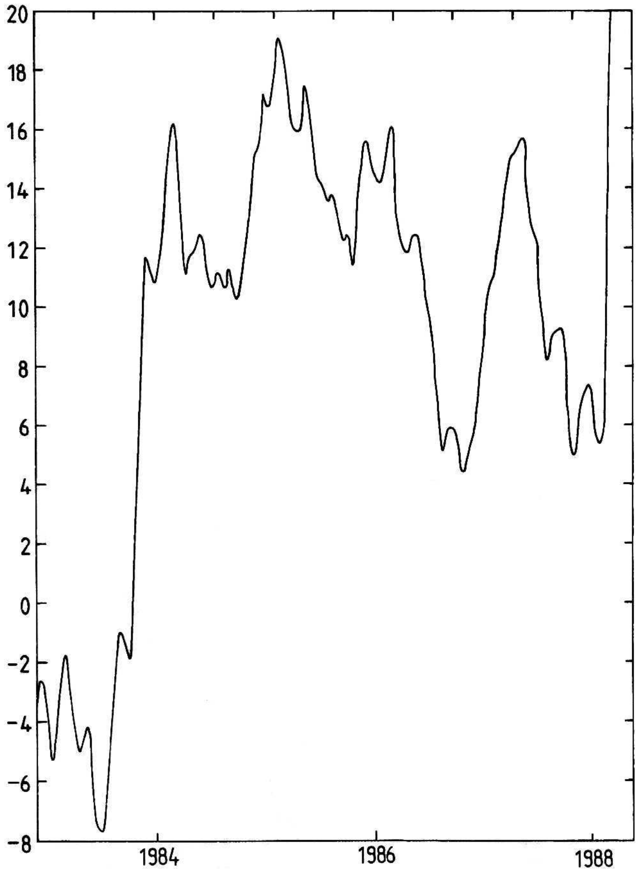

Fig. 6. The shift of the line of apsides of Mercury's orbit during five years only, calculated numerically from the N-body problem (adapted from (Rana, 1987)). The graph resembles deterministic chaos. The scale on the horizontal axis is given in years, and in the vertical axis the shift angle is in arc seconds.

The line segment connecting the perihelion and aphelion is called the *line of apsides*. As already mentioned, the orbital periods of the planets are not in a ratio of small integers, and therefore the line of apsides rotates quite irregularly. This follows from numerical simulations of the N-body problem. For instance, from a highly fluctuating and chaotic behavior of the shift of the line of apsides from Fig. 6 it is obvious that the value C from (4) cannot be determined with an accuracy better than several arc seconds per century. Therefore, in the literature several different values of C can be found, for instance,

$C = 532''$ per century (see (Misner et al., 1997, p. 1113)),
$C = 531''$ per century (see (Rydin, 2011, p. 1)),
$C = 530''$ per century (see (Vankov, 2010, p. 6)),

$C = 529''$ per century (see (Rana, 1987, p. 197)),

$C = 527''$ per century (see the last line of Table 1 taken from (Le Verrier, 1859)).

In spite of that the difference $E = O - C$ has only one expression $E = 43''$ per century in the currently available literature (see e.g. Table in (Pireaux, Rozelot, 2003, p. 1175)) and the Einstein value (3) is presented as accurate.

Moreover, it is not clear what is the definition of the perihelion shift per century. If it refers to some kind of average value, then we have to define exactly what average. Over which time period do we perform averaging? We will illustrate it on Fig. 6, where the time interval is only five years long. We observe that the perihelion shift may increase about $24''$ or decrease about $11''$ within less than one year, which has a nonnegligible influence on the average shift per century.

Let us point our that Fig. 6 corresponds only to a restricted N-body problem, where the Sun is fixed at the origin of heliocentric coordinates. In the real N-body problem (7), the Newtonian barycenter of the Solar system (see Fig. 3) is placed at the origin of Cartesian coordinates. These two problems have, of course, different modeling errors.

The inclination of Mercury's trajectory with respect to the ecliptic is $i = 7°$, and therefore, we do not deal with a planar problem (see Fig. 1). From Section 1, we know that positions of planets are calculated in rectangular heliocentric coordinates (X, Y, Z). Since the true solution of the N-body problem describing the Solar system is not known, it is approximated by finite series. For illustration we show how to parametrize, for instance, the X-coordinate of the Earth's center in time t:

$$
\begin{aligned}
X = {} & 0.0056114 + 0.001234\,t + 0.9998293\cos(1.7534857 + 6283.075850\,t) \\
& + 0.000011\,t\cos(2.02 + 6283.1\,t) \\
& + \sum_{i=1}^{38} A_i \cos(B_i + C_i t) + \sum_{i=39}^{40} A_i t \cos(B_i + C_i t),
\end{aligned}
$$

where the 120 constants A_i, B_i, C_i are given by a table in (Bretagnon, Francou, 1988, p. 313) and t is measured in thousands of years from the Julian date J2000.0. This is therefore an approximate analytical expression of X in the form of the sum of polynomials with trigonometric polynomials, which contributes to the total numerical error and violates the law of energy conservation. Similarly the remaining components Y and Z, and also the heliocentric coordinates of other planets are parametrized. Integration constants are corrected every few years so that they are consistent with the observed positions of the planets, cf. (Bretagnon, 1982) and (Bretagnon, Francou, 1988). However, in this way the difference between the observed and calculated perihelion shift is artificially reduced, i.e., the calculated trajectory of Mercury is not purely Newtonian. It is also not a numerical solution of (7), since approximations are corrected by observations.

Elliptical orbits of test particles can only be obtained for the central force field, which is proportional to the gravitational potential $1/r$. Nevertheless, this does not hold in the Solar system. For instance, the total weight of interplanetary dust around the Sun, which causes zodiacal light, is estimated to 10^{16} and 10^{17} kg. The paper (Roseveare, 1982) admits that its impact is almost insignificant on the Mercury's perihelion shift, together with the belt of asteroids, comets, etc. According to (Rydin, 2011), the solar oblateness (quadrupole moment) contributes to the overall perihelion shift of Mercury only $0.0254''$ per century (see also (Pireaux, Rozelot, 2003)). A somewhat larger value $3.4''$ per century is presented in (Weinberg, 1972, p. 200). The reason is perhaps that the length of the rotational axis of the Sun may oscillate. We should also take into account errors in the determination of physical constants (e.g. the gravitational constant G, the mass of the Sun and planets), the nonuniform influence of magnetic fields, tidal forces, etc. A large number of small errors can cause a nonnegligible total error $e_0 + e_1 + e_2$. On the other hand, Steven Weinberg [1972, p. 233] claims that only the Newtonian and the Einstein terms are large enough to be measured.

5. A method of Albert Einstein

If the general theory of relativity well describes the planetary motion, then the perihelion shift of Mercury calculated from Einstein's equations for the whole Solar system should be close to the observed shift O from (5). However, Einstein's equations for the Solar system cannot yield exactly the observed perihelion shift of Mercury O from (4), since every equation of mathematical physics is always only an approximation of reality and thus possesses a nonzero modeling error e_0 (see Fig. 4). Moreover, Einstein definitely did not solve his equations of general relativity for the Solar system, that are represented by a very complicated nonlinear system of partial differential equations. Their exact solution is not known even for two bodies, since the left-hand side contains several thousands of terms (partial derivatives of scalar functions). This is due to the fact that there are, in general, 10 independent components of the metric tensor, 20 independent components of the Riemann tensor, and 40 Christoffel symbols (for details see e.g. (Maeder, 2016), (Misner et al., 1997)).

Consequently, Einstein had to make a whole series of simplification to get some value of the perihelion shift of Mercury (see (3)). In other words, formula (2) has not been derived as a consequence of Einstein's equations in terms of mathematical implications.[2] On the other hand, it could give a good prediction, since the corresponding approximations were under control.

Einstein assumed that the curvature of space around the Sun is given by a time independent Schwarzschild metric outside a spherically symmetric object. That is, the considered curvature of space does not include the gravitational influence of Mercury, Jupiter, and other planets (cf. Fig. 3 and the right part of Fig. 2). Mercury is substituted by a test particle

[2] Mathematical implications are different from consequences in physics that may involve approximations. Let us present a trivial illustrative example: If $n = 8.992$ then n is not divisible by 3. Now, let us approximate n by $\tilde{n} = 9$. Then we cannot claim that $\tilde{n}$ is not divisible by 3.

with zero mass. The Schwarzschild metric is further replaced by a solution for the so-called weak gravitational fields, which uses the Parametrized Post-Newtonian formalism for velocities $v \ll c$. The fact that the speed of Mercury is considerably less than the speed of light enabled Einstein to perform a series of further simplification.[3] For example, Einstein neglected terms of higher order when calculating Christoffel symbols. After several pages (Einstein, 1915, p. 833–837) of other approximations he got an ordinary differential equation whose solution leads to the elliptic integral

$$\phi = [1 + \frac{\alpha}{2}(\alpha_1 + \alpha_2)] \int_{\alpha_1}^{\alpha_2} \frac{\mathrm{d}x}{\sqrt{-(x-\alpha_1)(x-\alpha_2)(1-\alpha x)}}, \qquad (8)$$

where $\phi > 180°$ is the angle between the radius-vector of perihelion and the radius-vector of aphelion of Mercury's orbit,

$$\alpha_1 = (a(1+e))^{-1}, \quad \alpha_2 = (a(1-e))^{-1},$$

and

$$\alpha = \frac{2GM}{c^2} \approx 3 \text{ km} \qquad (9)$$

is the Schwarzschild radius of the Sun. Because the integral in (8) has no known analytical solution, Einstein used the linear part of the Taylor expansion

$$\frac{1}{\sqrt{1-\alpha x}} = 1 + \frac{1}{2}\alpha x + \frac{3}{8}\alpha^2 x^2 + \dots$$

which is a fairly good approximation, since $\alpha\alpha_i \ll 1$ for $i = 1, 2$. Hence, the function ϕ from (8) was expressed as follows

$$\begin{aligned}\phi &\approx [1 + \frac{\alpha}{2}(\alpha_1 + \alpha_2)] \int_{\alpha_1}^{\alpha_2} \frac{(1 + \alpha x/2)\mathrm{d}x}{\sqrt{-(x-\alpha_1)(x-\alpha_2)}} \\ &= \pi\Big[1 + \frac{3}{4}\alpha(\alpha_1 + \alpha_2)\Big] = \pi\Big[1 + \frac{3\alpha}{2a(1-e^2)}\Big].\end{aligned}$$

From this, (9), and the 3rd Kepler[4] law $a^3/T^2 = GM/(4\pi^2)$ it follows that after one period (more precisely, between two successive perihelion passages) the perihelion shifts about the angle

$$\varepsilon = 3\pi\frac{\alpha}{a(1-e^2)} = 3\pi\frac{2GM}{ac^2(1-e^2)} = 24\pi^3\frac{a^2}{T^2c^2(1-e^2)},$$

[3] Their refinements are stated in the letter of Karl Schwarzschild to Albert Einstein dated 22. 12. 1915 – see the English translation of the article (Einstein, 1915).

[4] Note that all three Kepler's laws hold only approximately.

i.e. the relationship (2), which according to (3) yields the idealized value of the relativistic perihelion shift of Mercury 43″ per century (cf. the left part of Fig. 2).

It is curious that Einstein in [1915, p. 839] writes that astronomers observe also an additional perihelion shift of Mars' trajectory of 9″ per century, while his relationship (2) gives only 1″ per century.[5] This enormous discrepancy is not mentioned in the current literature. In other words, the questionable value for Mars is suppressed, whereas the value for the Mercury that fits to (1) is announced.

A number of publications show (see e.g. (Nobili, Will, 1986), (Quintero-Leyva, 2016), (Rydin, 2011)) the value of (3) even to four significant digits 42.98″ per century, when substituting more accurate values of a, e, and T to the simple algebraic equation (2). However, the fact that this relationship was derived by many approximations of Einstein's equations, authors usually do not comment, i.e., the errors e_0, e_1, and e_2 from Fig. 4 are ignored. Thus, the value of Mercury's perihelion shift (3) is not very reliable. Note that in (Ridao et al., 2014, p. 1712), the value $E = 42.9773350296''$ per century is presented, i.e., even to 12 significant digits!

6. Conclusions

In 1915, Albert Einstein derived the value of the additional relativistic perihelion shift of Mercury 43″ per century. In his article [1915, p. 831] he claims that Le Verrier needed approximately 45″ per century to explain the difference between the observed and calculated perihelion position. This caused a sensation and Einstein's general theory of relativity suddenly became famous. The value of relativistic shift (3), however, applies only to the idealized case, which may be quite different from reality. It was obtained by many approximations and assumptions. Consequently, the simple formula (2) should not be applied to strong gravitational fields.

It is very difficult to separate relativistic effects from effects of Newtonian mechanics of similar size and many other approximations. In formula (4), two almost equally large numbers are subtracted that are, in addition, perturbed by various errors. In other words, we do not evaluate the difference $O - C$, but $\tilde{O} - \tilde{C}$ without any guaranteed error estimates, where $\tilde{O} \approx O$ and $\tilde{C} \approx C$. The estimated difference of Mercury's perihelion shift obtained from astronomical observations and numerical solution of the problem of N-bodies is therefore ill-conditioned, see (Vankov, 2010). According to (Rydin, 2011), formula (2) represents a weak experimental and theoretical confirmation of general relativity.

[5] Substituting the values $a = 9377$ km, $e = 0.0151$, and $T = 7.65$ h corresponding to Phobos into (2), we can derive quite large value of the pericenter shift 23″ per century. This Martian moon is perhaps another suitable candidate for testing the validity of formula (2), since its trajectory is less disturbed by other bodies than Mercury. A secular trend in Phobos' longitude was already observed in (Lainey et al., 2007, p. 1082). For the most inner satellites of some other planets the formula (2) produces much higher pericenter shifts, e.g., for the Jupiter's moon Metis with $a = 127\,969$ km, $e \approx 0.04$, and $T = 7.075$ h we get $1.37°$ per century.

In this paper, we primarily wanted to point out that we should not indiscriminately take the value (3) and spread it further without checking how it was derived.

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References

Abbott B. P. et al., 2016, Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.* 116, 061102, pp. 1–16.
Anderson J. D. et al., 1996, Shape and orientation of Mercury from radar ranging data. *Icarus 124*, pp. 690–697.
Brandts J., Křížek M., Zhang Z., 2016, Paradoxes in numerical calculations. *Neural Netw. World* 26, pp. 317–330.
Bretagnon P., 1982, Théorie du mouvement de l'ensemble des planètes. Solution VSOP 82. *Astron. Astrophys.* 114, pp. 278–288.
Bretagnon P., Francou G., 1988, Planetary theories in rectangular and spherical variables. VSOP 87 solutions. *Astron. Astrophys.* 202, pp. 309–315.
Clemence G. M., 1947, The relativity effect in planetary motions. *Rev Modern Phys.* 19, pp. 361–364.
Einstein A., 1915, Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie. Königlich-Preußische Akad. Wiss. Berlin, pp. 831–839. English translation "Explanation of the perihelion motion of Mercury from general relativity theory" by R. A. Rydin with comments by A. A. Vankov, 1–34.
Goldberg D., 1991, What every computer scientist should know about floating-point arithmetic. *ACM Comput. Surveys* 23, pp. 5–48.
Inoue T., 1993, An excess motion of the ascending node of Mercury in the observations used by Le Verrier. *Celest. Mech. Dyn. Astron.* 56, p. 69.
Jurgens R. F. et al., 1998, Mercury radar ranging data from 1987 to 1997. *Astron. J.* 124, pp. 486–488.
Kolesnik Y. B., Masreliez C. J., 2004, Secular trends in the mean longitudes of planets derived from optical observations. *Astron. J.* 128, pp. 878–888.
Kopeikin S. M., Fomalont E. B., 2006, Aberration and the fundamental speed of gravity in the Jovian deflection experiment. *Found. Phys.* 36, pp. 1244-–1285.
Křížek M., 2015, Zdeněk Kopal — numerical analyst. *Astron. Soc. Pacific Conf. Ser.* 496, pp. 19–31.
Lainey V. et al., 2007, First numerical ephemerides of the Martian moons. *Astron. Astrophys.* 465, pp. 1075–1084.
Le Verrier U. J., 1859, Theorie du mouvement de Mercure. *Ann. Obser. Imp. Paris V* 76, pp. 1–196.
Maeder A., 2016, Scale invariant cosmology: cosmological models and small scale effects. Proc. Internat. Conf. Cosmology on Small Scales (eds. M. Křížek and Y. V. Dumin), Inst. of Math., Prague, pp. 41–64.
Misner C. W., Thorne K. S., Wheeler J. A., 1997, Gravitation, 20th edition. W. H. Freeman, New York.
Narlikar J. V., Rana N. C., 1985, Newtonian N-body calculations of the advance of Mercury's perihelion. *Mon. Not. R. Astron. Soc.* 213, pp. 657–663.
Newcomb S., 1895, The elements of the four inner planets and the fundamental constants of astronomy. Supp. Am. Ephemeris and Nautical Almanac for 1897, Washington, D.C., Gov. Printing Office, pp. 1–202.
Nobili A. M., Will C. M., 1986, The real value of Mercury's perihelion advance. *Nature 320*, pp. 39–41.
Quintero-Leyva B., 2016, On the intrinsic precession of the perihelion of planets of the Solar system. *Open Access Library J.* 3, e2554.

Pireaux S., Rozelot J.-P., 2003, Solar quadrupole moment and purely relativistic gravitation contributions to Mercury's perihelion advance. *Astrophys. Space Sci.* 284, pp. 1159–1194.
Pitjeva E. V., 2005, High-precision ephemerides of planets – EPM and determination of some astronomical constants. *Solar Sys. Res.* 39, pp. 176–186.
Pitjeva E. V., Pitjev N. P., 2014, Development of planetary ephemerides EPM and their applications. *Celest. Mech. Dyn. Astron.* 119, pp. 237–256.
Rana N. C., 1987, An investigation of the motions of the node and perihelion of Mercury. *Astron. Astrophys.* 181, pp. 195–202.
Ridao L. S. et al., 2014, Perihelion advances for orbits of Mercury, Earth, and Pluto from extended theory of general relativity (ETGR). *Can. J. Phys.* 92, pp. 1709–1713.
Roseveare N. T., 1982, Mercury's perihelion from Le Verrier to Einstein. Clarendon Press, Oxford.
Rydin R. A., 2011, The theory of Mercury's anomalous precession. Proc. of the NPA, vol. 8, College Park, MD, pp. 1–6.
Saari D. G., Xia Z., 1995, Off to infinity in finite time. *Notices Amer. Math. Soc.* 42, pp. 538–546.
Srinivasan D. K. et al., 2007, The radio frequency subsystem and radio science on the MESSENGER Mission. *Space Sci. Rev.* 131, pp. 557–571.
Standish E. M., 2004, An approximation to errors in the planetary ephemerides of the Astronomical Almanac. *Astron. Astrophys.* 417, pp. 1165–1171.
Standish E. M. et al., 2005, JPL planetary and Lunar ephemerides, DE403/LE403. JPL IOM 314.10-127, pp. 1–27.
Taganov I. N., 2016, Irreversible time physics. Russian Acad. Sci., Saint Petersburg.
Tisserand M. F., 1880, Les travaux de Le Verrier. *Ann. Observ. Paris* XV.
Titov O. A., 2011, Estimation of the acceleration of the Solar-system barycenter relative to a system of reference quasars. *Grav. Cosmol.* 55, pp. 95–99.
Vankov A. A., 2010, General relativity problem of Mercury's perihelion advance revisited. ArXiv: 1008.1811v1, pp. 1–46.
Weinberg S., 1972, Gravitation and cosmology: Principles and applications of the general theory of relativity. John Wiley & Sons, Inc., New York, London.