Titius-Bode law in the Solar System. Dependence of the regularity parameter on the central body mass

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Abstract. Near-commensurability of the orbital sizes or periods exists in the Solar system for the massive planets and the massive satellites of Jupiter, Saturn and Uranus. It is well revealed by the Titius-Bode law (TBL) long ago by Dermott (1968), but is not been explained convincingly yet. Independently on this fact, the question about the dependence of the scale constant of the TBL on the mass of the central body is open. In this paper we show such a dependence. Due to the dynamic evolution the orbits of the massive planets and satellites may be in a transient stage when a primary TBL is well pronounced. Simultaneously a secondary TBL, a trail from the past as a hint for the future, may be less pronounced. The TBL is fitted after the numeration of the objects. For this reason we derive a special "error curve" and we use 2 its minimums to introduce a primary and a secondary numeration for the objects. Thus we derive constants of 2 TBLs and build the searched dependence by twice as many points. In this paper we show and use pairs of TBLs for the satellite systems of Jupiter, Saturn, Uranus, Neptune and Pluto, as well as for the solar system in two cases – with 4 massive planets and with 8 massive planets. In fig. 10 we show the statistically significant dependences where the coefficient of the near-commensurability for the orbital sizes varies from about 1.3 for the satellites of Pluto to about 1.7 for the planets of the Sun.
Key words: Solar system: general – Solar system: structure

1.Introduction

For a long time now it has been recognized that the distribution of the orbit sizes of the solar planets and the planet satellites is non-random but near-commensurable. The orbit sizes (or periods) follow approximate power relations, known today as the Titius-Bode law (TBL, Eq. 2 and Eq. 3). Maybe it is not possible today to conclude with certainty that the TBL type relations are, or are not, significant, but the question about the physical explanation remains open. Still, the existence of TBL type relations give evidence about the combination of influences which has been formed during the past and continue to form the system of orbits in the Solar System (Dubrulle & Graner, 1994; Murray & Dermott, 1999; Lynch, 2003).

The contemporary theoretic considerations and computer simulations have shown that the orbital resonances of the giant planets create in the plane of the Solar system wide concentric rings where stable orbits may exist over long periods of time. It is clear also the TBL may be a result of the collapse of the rotating proto-planetary cloud with its axial symmetry and exponentially decreasing density in radial direction (Dubrulle & Graner, 1994; Kotliarov, 2008; Bovaird & Lineweaver, 2013).

Although deep analysis and deduction by theoretical reasons is missing, the TBL begins to play a new role. Today it is used for analyzing exoplanet systems and forecasting exoplanets which have not been observed yet (Poveda & Laram 2008; Bovaird & Lineweaver, 2013). The last being mentioned, astronomers performed TBL analysis of 68 multiple-exoplanet systems containing at least 4 objects and predicted 141 suitable orbits for exoplanets, 73 from interpolation, 68 from extrapolation.

Since the physical origin of the Titius-Bode relation has not been fully understood yet, we must ask whether the constant of the near-commensurability depends on the mass of the central body. The TBLs, derived long ago by Dermott (1968), show an increase of this constant using data from the satellites of Uranus throughout the satellites of Jupiter and the planets of the Sun. However, this constant for the satellites of Saturn is too small and it does not support such a trend.

Still, the dynamical evolution of every system of orbits causes changes of the system and we consider that every orbital system may always be in a transient epoch. Therefore, we must account not only for one optimal TBL, but for at least two, primary and secondary. We consider that the secondary TBL is a former or a future primary TBL. In this paper we find a poorly pronounced secondary TBL of the system of Saturn, which makes it compatible with the system of Jupiter. Furthermore, we build TBLs for the system of Neptune, which occurs to be compatible with the system of Uranus. We build also TBLs for the satellite system of Pluto and TBLs for the solar planet system. The resulting correlations, which are claimed in the title of this paper, are shown in Fig. 10.

In this paper Section 2 represents the historic Titius-Bode rule and its contemporary form, known as Titius-Bode law. Section 3 represents the method for objective numbering of the planets of the satellites for deriving primary and secondary TBLs. Section 4 represents the result based on the satellite systems of Jupiter, Saturn, Uranus, Neptune and Pluto, as well as on the Solar system in two cases: based on 4 regular planets or on 8 regular planets. Section 5 represents the result and Section 6 gives a summary of the presented work.

This investigation is based on contemporary data about orbital periods of 5 systems of planet satellites (Table 1, Sheppard, 2014), as well as of 8 large planets plus 9 dwarf planets (Table 2, Brown, 2012).

2. Titius-Bode Rule (TBR) and Titius-Bode Law (TBL)

The near-regularity of the planet orbits has been established initially by Johannes Titius ,Wittenberg, 1766. Later this fact has been confirmed and widely popularized by Johannes Bode, Berlin, 1772. Today this result is known as the Titus–Bode Rule (TBR): When the semi-major axis of the planetary orbit A is expressed in astronomical units (AU; 1 $AU = 149.6{\times}10^6$ km), then the TBR predicts A' by the relation

$$(1) \qquad A' = 0.4 + 0.3 \times 2^k.$$

Here the experimentally adopted constants correspond approximately to the size of the Mercury orbit (0.4 AU) and to the difference between the sizes of the Venus and Mercury orbits (0.3 AU). The power number k takes special values: $-\infty$ for Mercury, 0 for Venus, 1 for Earth, 2 for Mars, 4 for Jupiter and 5 for the last planet known at that time, Saturn.

Later, the orbit of Uranus, discovered by William Hershel, London, 1781, has confirmed the TBR for $k = 6$ and this fact has stimulated the further searches of planets. The vacancy between Mars and Jupiter for k=3 has been fulfilled by the asteroid Ceres, discovered by Giuseppe Piazzi, Palermo, 1801. However, the size of the orbit of Neptune, discovered by Johann Galle, Berlin, 1846, approves of 77 % of the TBR prediction for k=7. Further, the size of the orbit of Pluto, discovered by Clyde Tombaugh, Lowell Observatory, Arizona, 1930, approves of 61 % of the TBR prediction for k=8. Finally, the TBR has been rendered wrong.

The contemporary TBL for the semi-major axes of the orbits A has power or linear forms:

$$(2) \qquad A_n = A_1 \times (A_C)^n \qquad \text{or} \qquad B_n = B_1 + B_C \times n.$$

Here the numbers n ($n = 1, 2, \ldots, N$) belong to a preliminary adopted integer numeration of the objects and B_n =log A_n, while the intercept constant B_1 =log A_1 and the slope constant B_C=log A_C have to be determined for every system of orbits. In (Eq. 2). A_1 and B_1 correspond to the first object in the system, while A_C means the coefficient of near-commensurability in the linear scale and B_C denotes the step of near-equidistantly spacing in the logarithmic scale.

The accuracy of a TBL in the logarithmic scale may be characterized by the (sample) standard error σ (log A). It is a small positive quantity. In the linear scale the respective relative value, dex[σ(log A)], is a positive quantity slightly greater than 1. Therefore, the accuracy of the TBL may be expressed by its relative standard error, expressed illustratively in percentage:

$$(3) \qquad \sigma_R(A)[\%] = [10^{\sigma(logA)} - 1] \times 100.$$

Figure 1a represents a comparison between the TBR predicted semi-major axis A' (Eq. 1) and the real A in log-log coordinates. The traditional numeration of the planets, including Ceres, is implemented (see also Table 2). This regression has a very low standard error σ(log A) = 0.012 and respectively a relative error in the linear scale $\sigma_R(A)$ = 2.8 %. So, in spite of the rough constant values in (Eq. 1), TBR poses very high accuracy for the first 8 planets (including Ceres and Pluto). The TBR is correct only in this case and today this fact is considered an accident.

Figure 1b represents the TBLs (Eq.2) as linear regressions of log A over n. The regressions are derived for 8 of the solar planets together, as well as for 4 planets of the Earth group or for the 4 giant planets, separately. The regression errors are implemented. While the relative error for all 8 planets is 14.5 %, for the two planet group separately it is about 1.5 times smaller (10 % and 9.6 %, respectively). TBL for all 8 planets is described better by 2nd order regression with relative error of 11.4 % (as shown). Generally, the two groups of solar planets may be distinguished even in the diagram of the TBL by their different coefficients A_C. More details are given by Graner & Dubrulle (1994).

The semi-major axis A and the orbital periods P are related by Kepler's 3rd law: $A \propto P^{2/3}$ or Δ log$A$ = 2/3 $\times\Delta$ log P. By this reason the TBL may be rewritten for periods P too. All contemporary applications of the TBL are based on orbital periods P, because (i) the orbital periods of the planet satellite are primary and accurately derived data items and (ii) the rotational period of the central body supports such kind of TBL (Dermott, 1968).

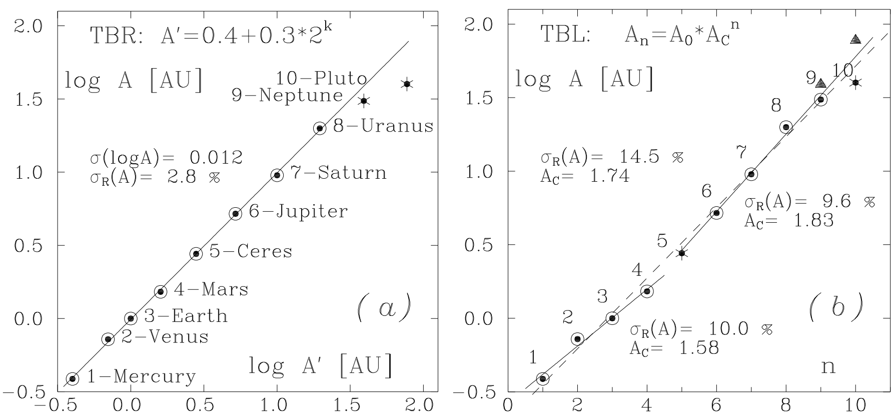

Fig. 1. (a) Original TBR in log-log coordinates. The line represents the regression of the real semi-major axes A over the TBR predicted ones A' (Eq. 1) for 8 planets (including Ceres). The real positions of Neptune and Pluto are presented by asterisks. (b) Contemporary TBL. The line represents the regressions of log A over the traditional orbital number n for the 8 regular planets together, without Ceres and Pluto (dashed line), as well as for the 4 large and for the 4 giant planets separately (solid lines). The positions of Ceres and Pluto are presented by asterisks. The position of Neptune and Pluto, predicted by the TBR, are represented by triangles.

The TBL written for the orbital periods P has power or logarithmic form:

$$(4) \qquad P_n = P_0 \times (P_C)^n \qquad \text{or} \qquad Q_n = Q_0 + Q_C \times n.$$

Here the numbers n ($n = 0, 1, 2, \ldots, N-1$) correspond to the preliminary adopted numeration of the objects, where number 0 is attached to the central body and Q_n = log P_n. The meaning of the constants Q_0 = log P_0 and Q_C = log P_C is analogous to the respective constants in (Eq. 2).

However, here P_0 corresponds to the rotational period of the central body. Then the respective semi-major axis of this "zero orbit" may be derived by the 3rd Kepler's law. For building the TBL the parameters Q_0 and Q_C have to be determined in every separate case. Because of the logarithmic forms of the TBL, written for A or for P, the relative regression errors in the linear scales are the same $\sigma_R(P) = \sigma_R(A)$.

The range and the appearance of the TBL depend crucially on the choice of the objects and the role of their numbers. Two curious examples follow.

Figure 2a shows TBL for a list of solar planets supplemented by Ceres, Pluto and the Sun (with its rotational period, as the object with number n=0). Two distant dwarf planets are added too: Eris, as the most massive and very distant plutoid, as well as Sedna, as the most distant and a massive enough plutoid. The range of the TBL here is 4 magnitudes by P, with a constant P_C=2.23 (corresponding to A_C=1.71) and a large error $\sigma_R(P) =$ 24.5 %.

Special form of the TBL for the solar planets has been proposed after analysis of the resonances in the planet periods by Carrey (1988). The "Carey's law" takes as base the orbital period of Jupiter, Jupiter's year, P_J, and assigns the number n=0 to Jupiter. The orbital periods of the more distant planets, with numbers 0, 1, 2, ..., 6, are given in Jupiter years [JY] as $P'_n = 3 \times P_J \times 2^{2n-1}$. Here the periods of the used planets, from Jupiter till Sedna,are 1, 3, 6, 12, 24, 48, 96 JY. The orbital periods of the closer planets, with numbers 0, -1, -2, -3,..., -6 are given in Jupiter years [JY] as $P'_n = 3 \times P_J \times 2^{-(n-1)}$. Here the periods of the used planets, from Jupiter till the Sun, are 1, 1/3, 1/6, 1/12, 1/24, 1/48, 1/96 JY.

Figure 2b shows the "Carey law" as a kind of TBL. In this case the relative error of the prediction of log P through log P' is $\sigma_R = 16.3$ %, about 1.5 times better than in Fig.2a. Like in the case of TBR (Eq. 1a.) the constants P_C or/and A_C can't be derived. The reasons for the large-scale near-commensurabilities, shown in Figs.2a dn 2b, have not been explained yet.

3. The method of optimal numbering of the orbital periods

This study, as well as all previous such ones, are based on "regular" planets or satellites. These objects must be preliminary selected and numbered.

While it is considered that in every case the TBL reflects the current result of the formation and evolution of the system, the TBL constants must be derived from "regular objects". These objects have large masses, as well as near-complanare with an equator of the central body and near-round orbits. Satellites and planets that are used here as regular objects are noted in Tables 1 and 2 by asterisks. Other significant distant objects, massive enough, even irregular are to be used here together with the regular ones as "all objects" for checking whether they support the TBL derived by the regular objects. Among irregular objects in Table 1 and 2 there are objects the orbits of which have high inclinations and high eccentricities, even objects with retrograde orbiting. Many other objects seem to be too small and they are to be neglected here.

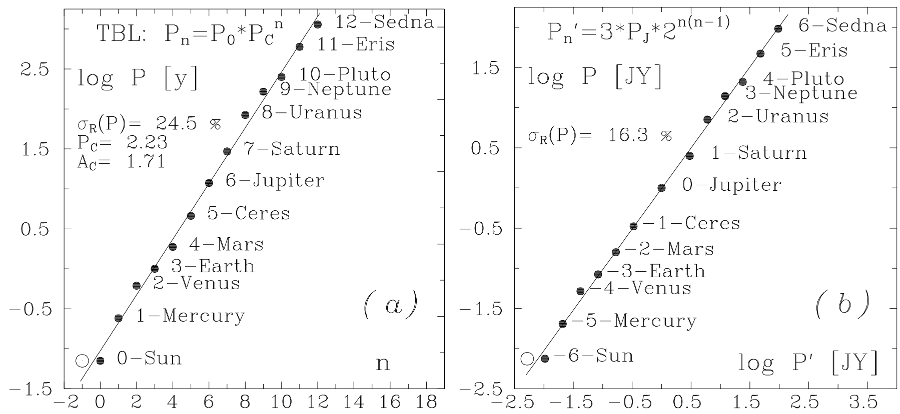

Fig. 2. Regression lines for of the TBL (a) and "Carey's law" (b) for the orbital periods of 13 objects of the Solar system (dots), were the Sun (with its rotating period) is included with n=0. The position of the Sun if its with preliminary givennumber isn=-1 is shown by circle.

Sometimes the requirement about regularity of the main objects is impracticable. One example is that the orbits of the 8 large solar planets are mutually near-complanare, but they are all significantly inclined with respect to the plane of the solar equator. We are forced to neglect this circumstance. Another example is that the Neptune satellite system is very poor and a direct building of a TBL is impossible. However, all relatively large retrograde satellites of Jupiter, Saturn and Uranus support the respective TBLs. Then including the retrograde satellite Triton we derive a TBL for Neptune system which is compatible with that of Uranus. A third example is the dwarf planet Pluto. It seems too small in comparison with the giant planets, however, the satellite system of Pluto follows well the TBL and supports well the general dependence, shown in Fig. 10.

The deriving of the constants of the TBL (4) depends significantly on the preliminary numeration of the objects. The examples in Fig. 1 and Fig. 2 show simple order numerations. One objective method for numeration is applied in the classic work of Dermott (1968). He built regressions of the type $n = a\times \log P + b$ for regular objects, deriving an optimal (integer) numeration for the regular objects and for the vacancies in between. This method is applied for the satellite systems of Jupiter, Saturn, Uranus and the planet system of the Sun. Thereby the problems about the application of the TBL appear to almost become closed. However, recently Bovaird & Lineweaver (2013) applied a χ^2 based method for optimal numeration of the members of exoplanet systems and revealing of vacancies among the exoplanets. In the presented work another approach to optimal numbering of the objects is applied. It is able to give many numerations, but here only "primary" and "secondary" such ones are used for deriving of "primary" and "secondary" TBL.

Let us have a few regular objects with orbital periods $P_n (n = 0, 1, \ldots, N-1)$.

The computer program scans possible interval of step constants Q_C, i.e. 0.1 $< Q_C <$ 0.6, with a small step, typically 0.001. For each checked value Q_C and for each object with number n the program derives a floating point "number" $\mu_n = (P_n - P_0)/Q_C$, the nearest to μ_n integer number m_n and the deviations $\Delta\mu_n = \mu_n \breve{} m_n$. Further the program derives for each Q_C the mean square deviation $E'(Q_C) = (\sum \Delta\mu - n2)/N$ and finds the mean value of these deviations $<E'(Q_C)>$ over all Q_C.

Finally, the program outputs the normalized "error curve" $E(Q_C) = E'(Q_C)/ < E' >$ with mean value equals to 1. The position of the two deepest minimum Q_{C1} and Q_{C2} of the error curve $E(Q_C)$ are considered as approximate primary and secondary step constants, that define the primary and secondary numerations M_1 and M_2. The regressions of the type (Eq. 4), based on the numerations M_1 and M_2 for regular objects (plus central body), give the accurate constants of the primary and secondary TBLs. The applications follow.

4. TBLs for satellites of the planets and for planets of the Sun

We applied the described method for optimal numbering on the satellite systems of Jupiter, Saturn, Uranus, Neptune and Pluto as well as twice on the solar planet system.

The results are presented in Fig. 3 – 9. The primary TBLs for the systems of Jupiter, Saturn and Uranus coincide with these described by Dermott (1968). The TBLs for the systems of Neptune and Pluto are added. The Solar System is the most complicated case and it is regarded in the end. Basic data about the satellites and the planets is given in Table 1 and Table 2.

The left panels of Fig. 3 – 9 show two error curves: $E(Q_C)(Q_C = logP_C)$, solid curves, derived through the regular objects and $E(Q_C)$, dashed curves, derived through all objects. The objects which are considered to be regular are marked in Table1 and Table 2 by asterisks.

The deepest minimums of the error curves $E(Q_C)$ for the regular objects with positions Q_{C1} and Q_2 are considered here to be primary and secondary minimums. Other minimums, at $\approx 0.5 \times Q_{C1}$ or $\approx 0.5 \times Q_{C2}$, are ignored as “undertones”. The deepness D_1 or D_2 of the minimums, as well as half the width and half the minimums W_1 or W_2, are determined visually and included in Table 3. The values of Q_{C1} and Q_{C2} are used by the computer program to give respective optimal numerations M_1 and M_2, shown in the last columns of Table 1 and Table 2.

Further these numerations are used for deriving TBL regressions of the type (Eq. 4). Their (sample) standard deviations σ_1 and σ_2 are given in Table 3. The slope coefficients of these regressions as the the accurate values of step constants Q_{C1} and Q_{C2} are marked by vertical lines on Fig. 3a – 9a. On the basis of these values other constants, P_{C1} and P_{C2}, B_{C1} and B_{C_2}, as well as A_{C1} and A_{C2} (see Eq. 2 and 4), are derived and included in Table 4. The relative errors of the TBLs, σ_{R1} and σ_{R2} (see Eq. 3), are given in Table 4 too.

The right panels of Fig. 3 – 9 show TBLs over the primary and secondary numerations, M_1 and M_2, corresponding to the steps Q_{C1} and Q_{C2}. The names of the objects and their numerations are implemented in the figures. In both cases the regression lines (solid and dashed, respectively) are built over the regular objects and prolonged to the edge objects. The presentation of the primary TBL corresponds to the abscissa numbers "m". For better illustration the position of the objects and their regression for the secondary TBL are shifted horizontally by -1 if $Q_2 > Q_{C1}$, with abscissa $\lfloor m - 1 \rfloor$ or by $+1$ if $Q_{C2} < Q_{C1}$, with abscissa $\lfloor m + 1 \rfloor$.

Jupiter system is presented in Fig. 3. This is the best example of near-commensurability in the Solar system. The Galilean satellites dominate in the Jupiter system and follow orbital period relations 1:2:4:≈8. Therefore, we must expect optimal step $Q \approx \log 2 = 0.301$. However, we include the rotating Jupiter as an additional object with number $n = 0$ and we find Q_{C1}=0.32 and Q_{C2} =0.20. The minimums at $Q_c = 0.16$ others left-situated ones are "undertones". Note that the primary minimum of $E(Q_C)$, found by the 4 large Gallilean satellites, is not supported well by all objects (dashed curve). Otherwise, like in all such studies, the primary TBL the Galilean satellites occupy their usual numbers of orbital periods (and semi-major axis) 2, 3, 4 and 5. In both cases Amalthea takes m=0, like Jupiter and Thebe takes m=1. These two inner satellites support the TBLs. Three vacancies occur after m=5. Two groups of relatively large distant satellites (one triad of irregular satellites and one triad of retrograde irregular satellites) take two distant orbital numbers. They support well both TBLs.

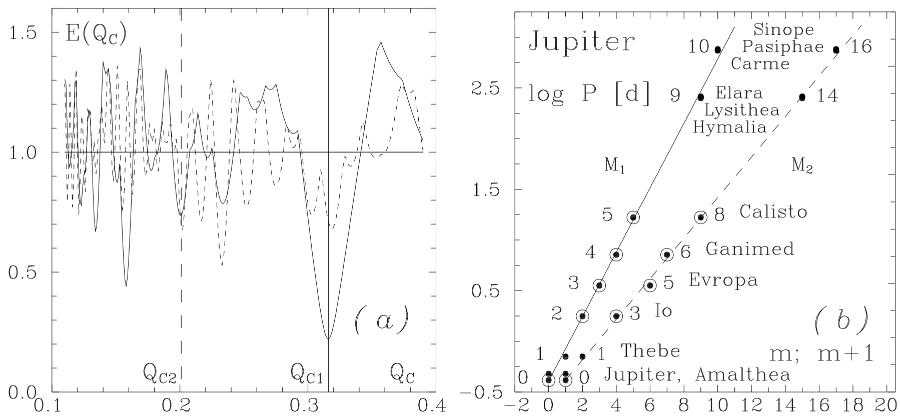

Fig. 3. Jupiter plus 4 regular satellites or plus all 12 significant satellites. See Table 1, 3, 4 and text. (a) Error curves $E(Q_C)$ for regular objects (solid curve) and for all objects (dashed curve). The positions of their primary (Q_{C1}) and secondary (Q_{C2}) minimums, for regular object, are marked by vertical lines. (b) TBL regressions for the regular objects (circles) by use of primary (M_1, solid line) and secondary (M_2), dashed line) numerations, prolonged to the end edge objects. The secondary TBL is shifted horizontally to the right by $+1$.

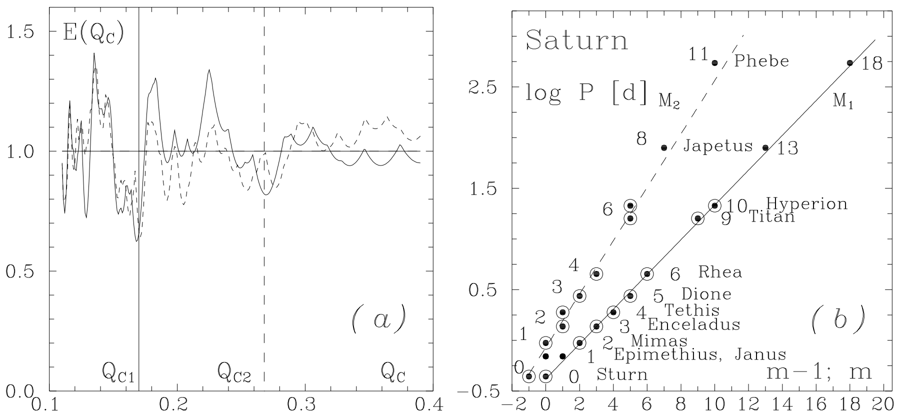

Fig. 4. Saturn plus 7 regular satellites or plus all 11 significant satellites. See Fig.3, Table 1, 3,4 and text. The secondary TBL is shifted horizontally to the left by -1.

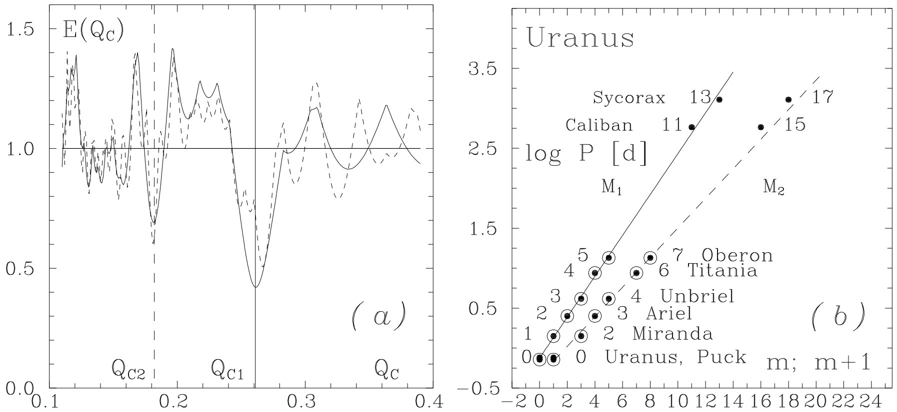

Fig. 5. Uranus plus 5 regular satellites or plus all 8 significant satellites. See Fig. 3, Table 1, 3, 4 and text. The secondary TBL is shifted horizontally to the right by +1.

Saturn system is presented in Fig.4. The scatter of the error curve $E(Q_C)$ is less than in the previous case because of the larger number of regular objects. Here the near-commensurability of the satellite orbits is supported by 7 regular satellites with Q_{C1}=0.17 and Q_{C2} =0.26. Two vacancies occur among the regular satellites in the primary TBL and one in the secondary. In the secondary numeration Mimas, Epimethius and Janus take m=1, Enceladus and Tetis take m=2, while Titan and Hyperion take m=6. Both TBLs are supported well by all used satellites. Only 2 irregular distant satellites are remarkable here - Japetus and (retrograde) Phoebe, which support well

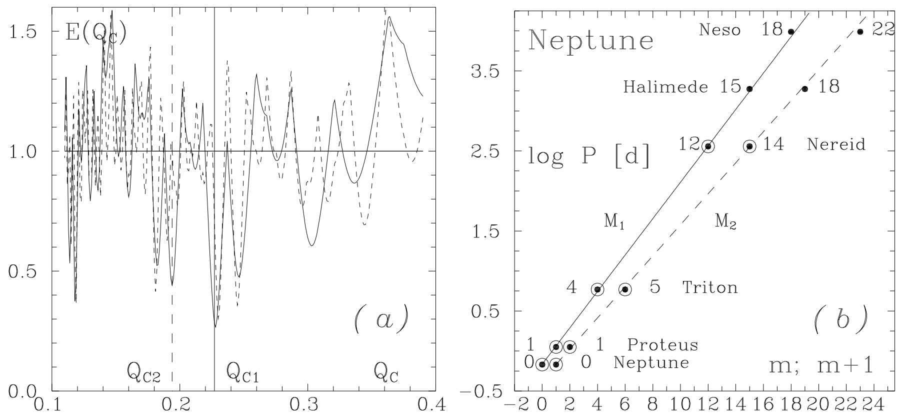

Fig. 6. Neptune plus 3 “regular” satellites or plus all 5 significant satellites. See Fig. 3, Table 1, 3, 4 and text. The secondary TBL is shifted horizontally to the right by +1.

the TBLs. In the system of Saturn the primary TBL corresponds to a very small (logarithmic) step Q_{C1} and this fact has been obstructive for finding the dependence of the TBL step on the mass of the central body. However, the poorly pronounced secondary minimum at Q_{C2}, elucidated here, softens this circumstance.

Uranus system is presented in Fig. 5. Here the near-commensurability is supported by 5 regular satellites. Like in the system of Jupiter, primary and secondary minimums of $E(Q_C)$ are well pronounced, here at Q_{C1}=0.23 and Q_{C2} =0.19. The inner satellite Puck take m=0, like Uranus. In the secondary TBLs one vacancy between Umbriel and Titania appears. Two distant satellites, both irregular and retrograde, support well the TBLs.

Neptune system is presented in Fig. 6. Here the regularity appears from 3 satellites, which include the large retrograde satellite Triton. We find Q_{C1}=0.26 and Q_{C2} =0.18. Two distant satellites, both irregular and retrograde, support well the TBLs. So, the TBLs of Neptune seems to be compatible with these of Uranus. Two distant satellites support well the TBLs.

Pluto system is presented in Fig. 7. Here the regularity appears again from 3 satellites, but the orbital period of the largest of them, Charon, is equal to the rotating period of Pluto. We found two deep minimums at Q_{C1}=0.20 and Q_{C2} =0.10. All known satellites support perfectly the secondary TBL. The system of Pluto occurs crucial for the correlations, shown in Fig. 10.

Solar planet system occurs more complicate than the satellite systems of the planets. Therefore, following the already applied logic, we regard firstly the case with only 4 regular planets (the giant ones), and secondly – the case with all 8 large planets.

Solar system with 4 regular planets, gas giants, is presented in Fig.8. We derive Q_{C1}=0.37 with A_C = 1.79 and Q_{C2} =0.43 with A_C = 1.93 (Table 4).Note that like in the case of Jupiter, the primary minimum of $E(Q_C)$,

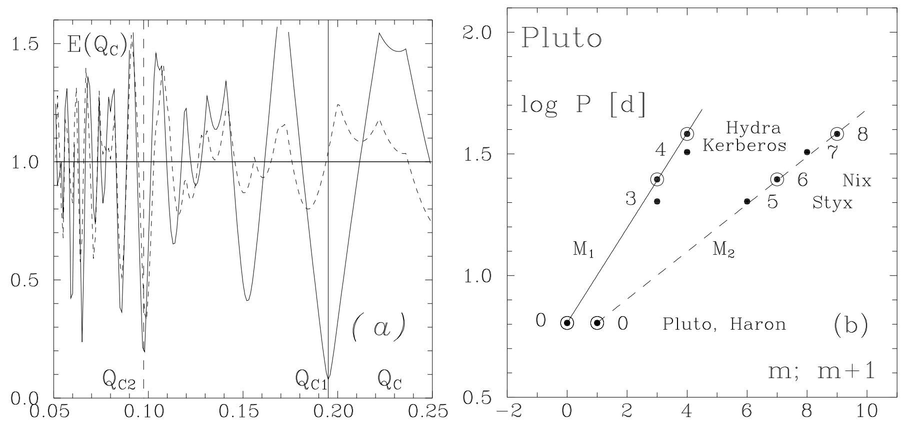

Fig. 7. Pluto plus 3 "regular" satellites or plus all 5 significant satellites. See Fig. 3, Table 1, 3, 4 and text. The secondary TBL is shifted horizontally to the right by +1.

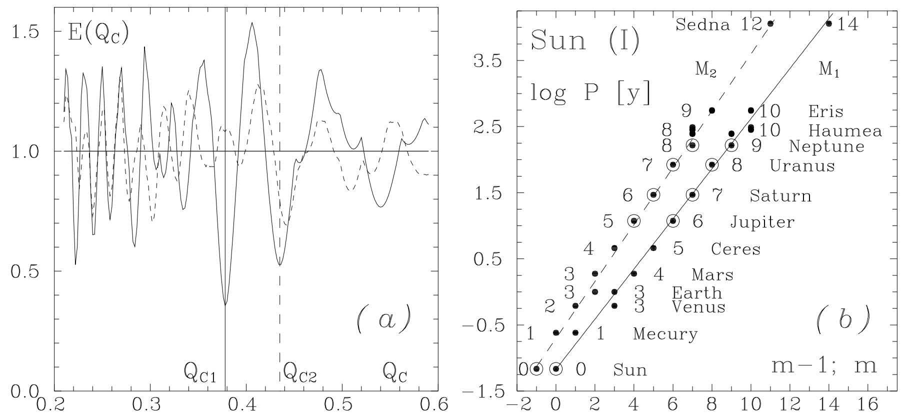

Fig. 8. Sun plus 4 regular planets (giant planets only) or plus 17 planets and dwarf planets. See Fig. 2, Table 2, 3,4 and the text. The secondary TBL is shifted horizontally to the left by -1. The numbers of Neptune, Haumea and Eris correspond to more than one object. See Table 2, columns M_1 and M_2.

found by 4 giant planets, is not supported by all objects (dashed curve). In the primary TBL the Earth and Venus take m=3, while m=2 corresponds to a vacancy. Then Neptune, Orcus and Pluto take m=9 and 5 other plutoids take m=10 (Table 2, column M_1). In the secondary TBL the Earth and Mars take m=3 with no vacancies around them. Then Neptune and 5 plutoids take m=8 and 2 other plutoids take m=9 (Table 2, column M_2). In both cases Sedna has a very distant and solitary objects.

Solar system with 8 regular planets is presented in Fig. 9. We derive Q_{C3}=0.24 with A_C=1.44 and Q_{C4}=0.28 with A_C=1.54. Here the primary and secondary minimum of $E(Q_c)$, found by 8 regular planets (solid curve),

Table 1. Basic data about the used satellites of planets from Sheppard (2014): D – diameter in km; P – orbital period in days; i - orbital inclination in respectto the equatorial plane of the planet; e –orbital eccentricity; M_1 and M_2 – nume-rations with respect to the primary and secondary minimum of the error curve $E(Q)$. Satellites accepted as regular are underlined. Retrograde periods of satellites are marked by "-".

	Label	Name	D[km]	P[d]	i[o]	e	M_1	M_2
Jupiter system								
1	V	Amalthea	167	0.479	0.374	.0032	0	0
2	XIV	Thebe	98	0.706	1.076	.0175	1	1
3	I	*Io	3660	1.769	0.050	.0041	2	3
4	II	*Europa	3122	3.551	0.471	.0094	3	5
5	III	*Ganymede	5262	7.155	0.204	.0011	4	6
6	IV	*Callisto	4820	16.689	0.205	.0074	5	8
7	VI	Himalia	170	250.23	30.486	.1513	9	14
8	X	Lysithea	36	259.89	27.006	.1322	9	14
9	VII	Elara	86	257.62	29.691	.1948	9	14
10	XI	Carme	46	-763.95	165.047	.2342	10	16
11	VIII	Pasiphae	60	-739.80	141.80	.3743	10	16
12	IX	Sinope	38	-739.33	153.778	.2750	10	16
Sarurn system								
1	XI	Epimetheus	116	+0.694	0.335	.0098	1	1
2	X	Janus	179	+0.695	0.165	.0068	1	1
3	I	*Mimas	396	+0.942	1.566	.0202	1	2
4	II	*Enceladus	504	+1.370	0.010	.0047	2	3
5	III	*Tethys	1062	+1.888	0.168	.0001	2	4
6	IV	*Dione	1123	+2.737	0.002	.0022	3	5
7	V	*Rhea	1527	+4.518	0.327	.0013	4	6
8	VI	*Titan	5151	+15.945	0.348	.0288	6	9
9	VII	*Hyperion	270	+21.277	0.568	.1231	6	10
10	VIII	Iapetus	1469	+79.322	15.47	.0286	8	13
11	IX	Phoebe	213	-545.09	173.047	.1562	11	18
Uranus system								
1	XV	Puck	162	0.762	0.319	.0001	0	0
2	V	*Miranda	471	1.414	4.232	.0013	1	2
3	I	*Ariel	1158	2.520	0.260	.0012	2	3
4	II	*Umbriel	1169	4.144	0.205	.0039	3	4
5	III	*Titania	1577	8.706	0.340	.0011	4	6
6	IV	*Oberon	1523	13.463	0.058	.0014	5	7
7	XVI	Caliban	72	-579.50	139.885	.1587	11	15
8	XVII	Sycorax	150	-1283.4	152.456	.5224	13	17
Neptune system								
1	VIII	*Proteus	420	1.122	0.08	.0005	1	1
2	I	*Triton	2705	-5.877	156.86	.0000	4	5
3	II	*Nereid	400	360.13	7.09	.7507	12	14
4	IX	Halimede	62	-1879.08	112.90	.265	15	18
5	XIII	Neso	60	-9740.73	131.26	.5714	18	22
Pluto system								
1	I	*Charonv	1208	6.387	0.00	.0022	0	0
2	V	Styx	20	20.162	0.81	.0058	3	1
3	II	*Nix	440	24.855	0.13	.0020	3	2
4	IV	Kerberos	31	32.168	0.39	.0033	4	3
5	III	*Hydra	50	38.202	0.24	.0059	4	4

is well supported by all objects (dashed curve). Here the distributions of the numerations M3 and M4 open many vacancies (see Table 2, columns M_3

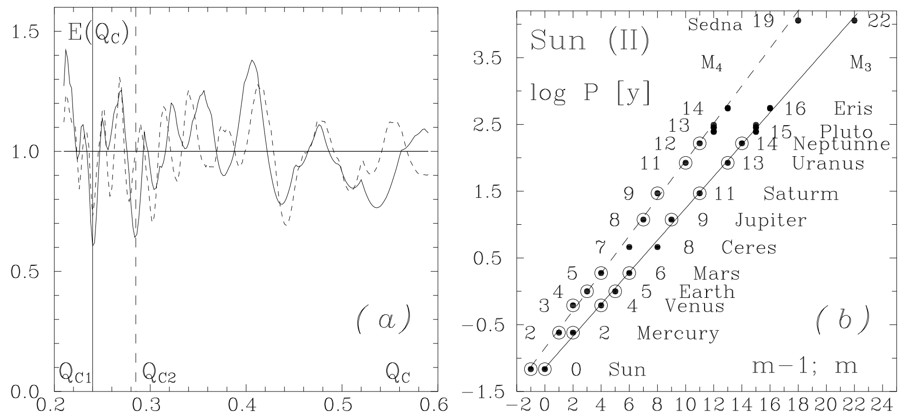

Fig. 9. Solar platurn system with 8 regular planets. See Fig. 3, Table 2, 3, 4 and text. The secondary TBL is shifted horizontally to the left by -1. The numbers of Pluto, Haumea and Eris correspond to more than one object. See Table 2, columns M_3 and M_4.

and M_4). Really the solar planet system occurs very complicated and we will use Q_{C3} on the diagrams in Fig. 10 as an additional primary minimum.

Table 2. Basic data about the 8 large planets and the 9 dwarf planets from Brown (2012): D – diameter in Earth diameters; P – orbital period in Earth years; i - orbital inclination in respect to the Earth orbit; e – orbital eccentricity; M_1 and M_2 – numerations in respect to the primary and secondary minimum of the error curve $E(Q)$ when only the 4 giant planets are used as regular planets (Fig. 8); M_3 and M_4 - numerations in respect to the primary and secondary minimum of the error curve $E(Q)$ when all 8 planets are used as regular planets (Fig. 9).

	Name	D[ED]	P[EY]	i[°]	e	M_2	M_1	M_3	M_4
1	*Mercury	0.382	0.241	7.0	.216	1	1	2	2
2	*Venus	0.949	0.615	3.4	.007	2	3	3	4
3	*Earth	1.0	1.0	0.0	.017	3	3	4	5
4	*Mars	0.532	1.881	1.85	.093	3	4	5	6
5	Ceres	0.076	4.60	10.59	.079	4	5	7	8
6	*Jupiter	11.209	11.862	1.30	.048	5	6	8	9
7	*Saturn	9.449	29.458	2.48	.054	6	7	9	11
8	*Uranus	4.007	84.018	0.77	.047	7	8	11	13
9	*Neptune	3.883	164.780	1.77	.009	8	9	12	14
10	Orcus	0.062	245.18	20.57	.227	8	9	13	15
11	Pluto	0.180	248.09	17.14	.249	8	9	13	15
12	Haumea	0.094	283.28	28.22	.195	8	10	13	15
13	Quaouar	0.070	285.97	8.00	.039	8	10	13	15
14	Make2	0.111	550.98	30.70	.500	9	10	14	16
15	2007-OR10	0.100	550.98	30.70	.500	9	10	14	16
16	Eris	0.182	557.00	44.19	.442	9	10	14	16
17	Sedna	0.076	11400.0	11.93	.853	12	14	19	22

5. Dependence of the logarithmic step of the orbital period on the logarithm of the central body mass

Figure 10 represent the correlations between the mass of the central body M (expressed in Eath mases, EM), in logarithmic scale and the constants Q_C= log P_C or B_C=log A_C (step of near-equidistantly distribution of the orbital periods or sizes), left panel or P_C or A_C (coefficients of near-commensurability of the orbital periods or sizes), right panel. The coefficients of correlation for both systems of data are about 0.7. The regression slopes are significant above the 99 % criterion of Student.

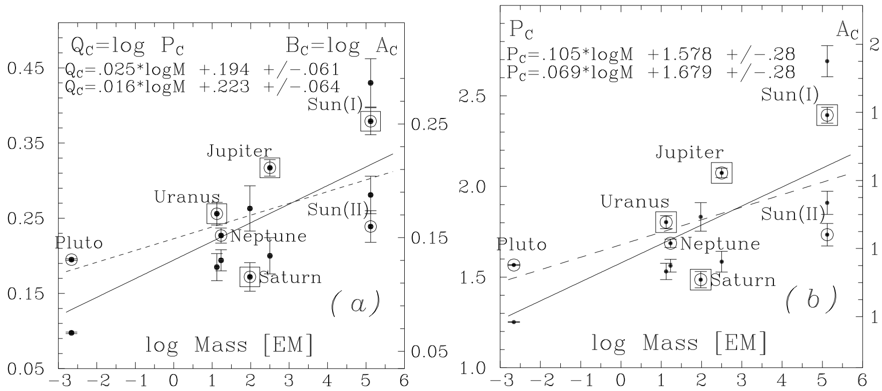

Fig. 10. Correlation between the logarithm of mass of the central body M (in Earth masses, EM) and the parameter of the regularity: primary (dots inside circles) and secondary (dots only). The solar planet system is presented by two pairs of constants: for 4 regular planets and for 8 regular planets (Table 4). Squares note 4 points that are known early from the paper of Dermatt (1968). Solid regression lines describe only primary TBL slopes and dashed lines describe all slopes. (a) The parameters are log $P_C = Q_C$ (left ordinate) or log A_C (right ordinate). (b) The parameters are P_C (left ordinate) or A_C (right ordinate).

In Fig.10 solid lines represent the regressions over primary parameters while dashed lines represent the regressions over all parameters. In all cases the regression weights of the data items are equal to 1. The respective weighted regressions, when the weights of the data items are proportional to the deepness D of the minimums (Table 3, columns D_1 and D_2), the regressions show the same appearance, being shifted upward by about 0.01.

6. Summary and conclusions

Four points on the diagrams in Fig.10, known by Dermatt (1968) and noted by squares, are not enough for defining of a good dependence.

In the presented work we suppose that the current status of any orbital system is result of an ongoing dynamical evolution and therefore transient.

Table 3. Parameters of the central body and minimums of the error curve: M - mass in kilograms; P – rotational period in days; N – number of used regular satellites of planets; D – deepnes of the minimum, W – size of the minimum at the half of its deepness; σ – standard error of the TBL (Eq. 4, in the logarithmic scale). Subscripts "1" or "2" indicate primary or secondary minimums.

	log M[kg]	P[d]	N	D_1	W_1	σ_1	D_2	W_2	σ_2
Jupiter	2.5	.413	4	.78	.041	.022	.27	.005	.048
Saturn	1.98	.440	7	.38	.018	.038	.19	.006	.065
Uranus	1.23	.719	5	.73	.011	.020	.56	.009	.029
Neptune	1.12	.670	3	.58	.014	.031	.31	.009	.037
Pluto	-2.66	6.387	3	.92	.005	.001	.80	.002	.071
Sun	5.12	25.05	8	.39	.011	.043	.35	.014	.051

Table 4. Final results about the parameters of TBLs in the Solar System:Q_C – step of the period growing in logarithmic scale; P_C – coefficient of period growing in linear scale; B_C and A_C – respective step and coefficient of growing of the semi-major orbital axes; σ_R – relative standard square error of the TBL. Here Q_C=logP_C (Eq. 4) and B_C=logA_C in (Eq. 2). Subscripts "1" or "2" indicate primary or secondary minimums.

	Q_{C1}	P_{C1}	B_{C1}	A_{C1}	σ_{R1}	Q_{C2}	P_{C2}	B_{C2}	A_{C2}	σ_{R2}
Jupiter	.317	2.07	.211	1.63	5.2	.200	1.58	.133	1.36	11.7
Saturn	.172	1.49	.115	1.30	9.1	.263	1.83	.175	1.50	16.1
Uranus	.227	1.69	.151	1.42	4.7	.194	1.56	.129	1.35	6.9
Neptune	.256	1.80	.171	1.48	7.4	.185	1.53	.123	1.34	8.9
Pluto	.195	1.57	.130	1.35	2.3	.0975	1.25	.065	1.16	2.3
Sun(I)	.379	2.39	.253	1.79	8.8	.430	2.69	.287	1.93	17.7
Sun(II)	.239	1.73	.159	1.44	10.4	.281	1.91	.187	1.54	12.4

So, by excepting a single well prominent parameter of near-regularity (a coefficient in the linear scale or a step in the logarithmic scale), called here primary parameter, another poorly prominent such parameter, corresponding to a past or a future status of the system may be found. For that reason we proposed a method that scans the possible values of the parameter of near-regularity and leads us to parameters of primary and secondary TBLs. We apply this method for the regular satellites of Jupiter, Saturn, Uranus, Neptune and Pluto, as well as for four or eight planets of the Solar System. The derived regularity parameters (primaries only, or both, primaries and secondaries) increase with the increasing of the mass of the central body. The dependencies, which cover 99 % Student's test, are shown in Fig. 10.

It is interesting whether such a dependence is supported by the orbits of the exoplanet systems.

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