Three-planet resonances in the Solar system

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Abstract. To be able to explain more precisely the structure, origin and evolution of the Solar system we need to know more relations between orbital elements of the bodies in the Solar system. The aim of this work is to find relations between the synodic periods of the bodies in the Solar system, so that their ratio is a simple integer ratio, and to ask: Are they just a coincidence or is their origin a result of the evolution of the Solar system? Some relations between mean longitudes of the bodies in the Solar system that are similar to relationship between the mean longitudes of the Jupiter's satellites Io, Europa and Ganimed are also derived. The reason of the existence of some of them is explained. Further particular solutions of the four-body problem using the perturbation theory approach could explain the nature of others. A period of configuration repetition of three planets is defined. It is named a synodic (gr. $\sigma\upsilon\nu o\delta$ - companion) period of three-planets because if the three planets are initially in a line from one side of the Sun, i.e. closest to each other, after this period they will be "companions" again. All the planets in the Solar system, the dwarf planets Ceres and Pluto, the first four discovered asteroids: 2 Pallas, 3 Juno, 4 Vesta, 5 Astraea and main resonant asteroids with Jupiter: 279 Thule (3:4), 153 Hilda (2:3), 108 Hecuba (1:2), 1362 Griqua (1:2), 8 Flora (2:7), 887 Alinda (1:3), 434 Hungaria (1:4) are objects of research in the current study and all of them are involved in three-planet resonances. In general the three-planet resonances are more numerous and more accurate than the two-planet orbital resonances. It was found that there are relations between the mean motions of four bodies. These four-planet resonances will be subject to a further study. Our initial calculations show that there is a relation between the mean motions of all terrestrial planets, dwarf planet Ceres and Jupiter and there is a relation between the mean motions of all giant planets and dwarf planet Pluto. More detailed study of such resonances could help to explain the structure and the origin of the Solar system.
Key words: mean motion resonance, three-planet resonance, synodic period, mean longitude, syzygy, Solar system

Introduction

There are different kinds of resonances in the Solar system. Two-planet mean motion orbital resonances are the resonances Jupiter - Saturn (2:5) (Great inequality) [2001, Michtchenko & Ferraz-Mello], Hilda - Jupiter (3:2), Hecuba - Jupiter (2:1) and Neptune - Pluto (3:2) [1976, Peale]. A spin orbital resonance occurs when the ratio of the orbital period and the rotation period of a body is a simple integer ratio. Such resonances are: Moon - Earth (1:1), Io - Jupiter (1:1) and Mercury - Sun (3:2). There is a relation between the Venus rotation period, its orbital period and the orbital period of the Earth [1967, Goldreich & Peale]:

$$\frac{4}{P_V} - \frac{5}{P_E} + \frac{1}{P_r} = 0, \tag{1}$$

where $P_r = -243\ dy$ is the Venus rotation period and P_V and P_E are the orbital periods of Venus and Earth respectively. In consequence of this dependence at each Venus conjunction the same side of Venus is facing the Earth.

In 1619 Kepler wrote about the resonance structure of the Solar system [1969, Gingerich], [1939, Kepler], [1997, Kepler & al.]. In "Orbital resonances in the Solar System" [1976, Peale] Peale explores the main resonances. A classic example of a three-planet resonance is the motion of the Galilean satellites of Jupiter: Io, Europa and Ganymede. It is named Laplace resonance. Europa's orbital period is approximately twice Io's period and Ganymede's orbital period is approximately twice Europa's period. The exact relation is:

$$\frac{1}{P_{Io}} - \frac{3}{P_{Eu}} + \frac{2}{P_{Ga}} = 0. \tag{2}$$

It can be written as:

$$P_{EuGa} = 2P_{IoEu}. \tag{3}$$

I.e. the synodic period of Europa and Ganymede P_{EuGa} is twice the synodic period of Io and Europa P_{IoEu}.

There is a similar correlation between the orbital periods of Venus, Earth and Mars [2011 a, Wilson]:

$$\frac{3}{P_V} - \frac{7}{P_E} + \frac{4}{P_{Ma}} \approx 0. \tag{4}$$

The last equation shows that the ratio of the synodic periods of Venus and Mars is approximately equal to 3/4. If Venus, Earth and Mars form a configuration, then after a time equal to four Venus synodic periods (6.4 yr), they will form the same configuration with a deviation of 1°. The mismatch time[1] is 1000 yr.

Other relations between the mean motions of the planets are known [1968, Molchanov]:

$$\begin{aligned} n_{Ma} - 6n_J - 2n_U &\approx 0, \\ n_V - 3n_{Ma} - n_S &\approx 0, \\ n_S - 5n_N + n_{Pl} &\approx 0. \end{aligned} \tag{5}$$

In "The reality of resonances in the solar system" Molchanov continues his works on resonances [1969, Molchanov] [1969, Gingerich]. In "Resonant Structure of the Outer Solar System in the Neighborhood of the Planets" [2001, Michtchenko & Ferraz-Mello] the stability of the Solar system is examined, using following relation between the mean motions of the giant planets Jupiter, Saturn and Uranus:

$$3n_J - 5n_S - 7n_U \approx 0. \tag{6}$$

[1] time for the rotation of the middle body by 180° to the initial configuration

In "The web of three-planet resonances in the outer Solar System" Massimiliano Guzzo [2005, Guzzo] numerically detects the web of three-planet resonances (i.e., resonances among mean anomalies, nodes and perihelia of three planets) with respect to the variation of the semi-major axis of Saturn and Jupiter. In "The Role of Resonances in Astrodynamical Systems" Rudolf Dvorak [2010, Dvorak] explores relations between the mean motions of asteroids and the giant planets Jupiter and Saturn.

In this article we look for relations similar to equations (2) and (4) between the orbital periods of the bodies in the Solar system. All the planets in the Solar system, the dwarf planets Ceres and Pluto, the first four discovered asteroids: 2 Pallas, 3 Juno, 4 Vesta, 5 Astraea and the main resonant asteroids with Jupiter: 279 Thule (3:4), 153 Hilda (2:3), 108 Hecuba (1:2), 1362 Griqua (1:2), 887 Alinda (1:3), 8 Flora (2:7?), 434 Hungaria (1:4) are included here. The resonance Flora - Jupiter is controversial, but this asteroid is included as it is considered that the Flora family consists of more than 5000 members and it represents 4-5% of all main-belt asteroids.

Table 1 shows the mean motions of the planets and the dwarf planets Ceres and Pluto.

Table 1. Mean motions of the terrestrial planets and Ceres

Planet	Mercury	Venus	Earth	Mars	Ceres
$n[^\circ/dy]$	4.092338799	1.602130477	0.985609113	0.524032836	0.214328175
Planet	Jupiter	Saturn	Uranus	Neptune	Pluto
$n[^\circ/dy]$	0.083091189	0.033459652	0.011730787	0.005981826	0.00397846

Table 2 and Table 3 give the mean motions of the considered asteroids. The values for the planets are taken from "Astronomical yearbook" [1987,Astronomicheskij Ezhegodnik] and for the asteroids and the dwarf planets Ceres and Pluto - from Jet Propulsion Laboratory (JPL) [JPL Small-Body Database Browser].

Table 2. Mean motions of the first four discovered asteroids and 8 Flora

Asteroid	2 Pallas	3 Juno	4 Vesta	5 Astraea	8 Flora
$n[^\circ/dy]$	0.213539504	0.225761927	0.271524121	0.238688795	0.363520232

Table 3. Mean motions of main resonant asteroids with Jupiter

Asteroid	434 Hungaria	887 Alinda	1362 Griqua	108 Hecuba	153 Hilda	279 Thule
$n[^\circ/dy]$	0.301827948	0.252689529	0.17083233	0.16878079	0.12451537	0.11160121

1 A mathematical approach for finding a three-planet resonance

Let us consider three bodies with orbital periods P_1, P_2, P_3 and $P_1 < P_2 < P_3$. P_{12} is the synodic period of the first and the second body:

$$\frac{1}{P_{12}} = \frac{1}{P_1} - \frac{1}{P_2} \tag{7}$$

and P_{23} is the synodic period of the second and the third body:

$$\frac{1}{P_{23}} = \frac{1}{P_2} - \frac{1}{P_3}. \tag{8}$$

There is a three-planet resonance if the ratio of the synodic periods P_{12} and P_{23} is a simple integer ratio:

$$\frac{P_{12}}{P_{23}} = \frac{i}{j}, \qquad i, j = 1, 2, 3, ... \tag{9}$$

The last equation can be written as:

$$in_1 - (i + j)\, n_2 + jn_3 = 0, \tag{10}$$

where n_1, n_2, n_3 are the mean motions of the planets.

Because $P_1 < P_2 < P_3$, the period of configuration repetition P_{ij} of the three planets can be derived from:

$$P_{ij} = \frac{360^\circ\,(i + j)}{n_1 - n_3} = (i + j)\, P_{13}, \qquad i, j = 1, 2, 3, ... \tag{11}$$

where P_{13} is the synodic period of the first and the third planets. The period P_{ij} will be named synodic (gr. $\sigma\upsilon\nu o\delta$ - companion) because if the three planets are initially in a line from one side of the Sun, i.e. closest to each other, after this period they will be "*companions*" again.

Let the three planets lay in one line with the Sun (Fig.1). The deviation of the mean longitudes of the planets φ for time equal to the synodic period P_{ij} is calculated from:

$$\varphi = \Delta\lambda_1 - \Delta\lambda_2, \tag{12}$$

where $\Delta\lambda_1$ and $\Delta\lambda_2$ are the changes of the mean longitudes of the first and the second planet respectively.

The deviation per year ψ can also be taken to determine the accuracy of the resonance:

$$\psi = \frac{\varphi}{P_{ij}\,[yr]}, \qquad i, j = 1, 2, 3, ... \tag{13}$$

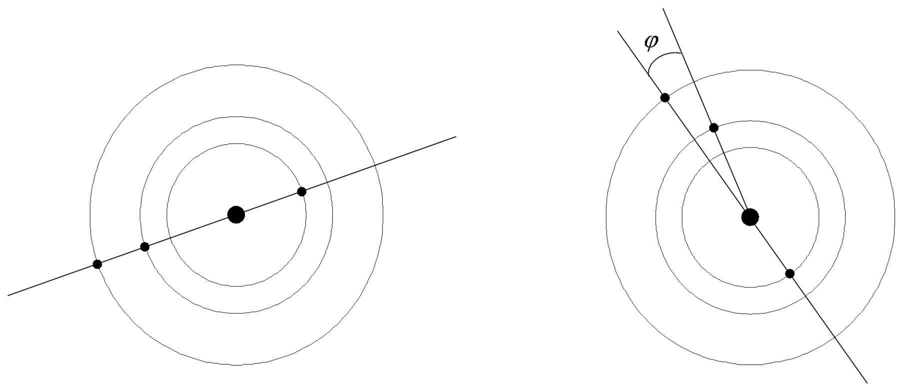

Fig. 1. Initial configuration of the tree planets (left) and configuration after the synodic period P_{ij} (right)

The time of mismatch P_m can be derived from:

$$P_m = \frac{180^\circ P_{ij}}{\varphi}, \qquad i, j = 1, 2, 3, ... \tag{14}$$

The relation (10) is approximately true. We can write an exact equation:

$$\frac{i}{P_1'} - \frac{(i+j)}{P_2} + \frac{j}{P_3} = 0, \qquad i, j = 1, 2, 3, ... \tag{15}$$

where P_1' is approximately equal to P_1:

$$P_1' = P_1 + \Delta P_1 \tag{16}$$

and ΔP_1 is a small value. The quantity $\Delta P_1/P_1$ can also be used to determinate the accuracy of the resonance between the tree planets. In this work the maximum value of $|\Delta P_1|/P_1$ is 1%.

We introduce two more small parameters to determine the accuracy of the resonance:

$$\frac{in_1 - (i+j)\, n_2 + jn_3}{n_1 - n_3},$$

$$\frac{iP_{23} - jP_{12}}{(i+j)\, P_{13}}.$$

Here three-planet resonances are calculated using the following algorithm:

1. Input n_1, n_2, n_3.
2. Calculate n_{12}/n_{23} from:

$$\frac{n_{12}}{n_{23}} = \frac{n_1 - n_2}{n_2 - n_3}. \tag{17}$$

3. Check $n_{12}/n_{23} > 1$.
4. Input the index $i = 1$. (If $n_{12}/n_{23} \leq 1$ the index j is input.)
5. Calculate the index j from:

$$j = i \left[\frac{n_{12}}{n_{23}} \right], \tag{18}$$

   where $[n_{12}/n_{23}]$ is the closest integer number approximately equal to the ratio n_{12}/n_{23}.
6. Calculate P_{ij}, φ, ψ, P_m, ΔP_1 and $\Delta P_1/P_1$.
7. Input a new value of the index i, $i := i + 1$ and repeat the process from step 5.

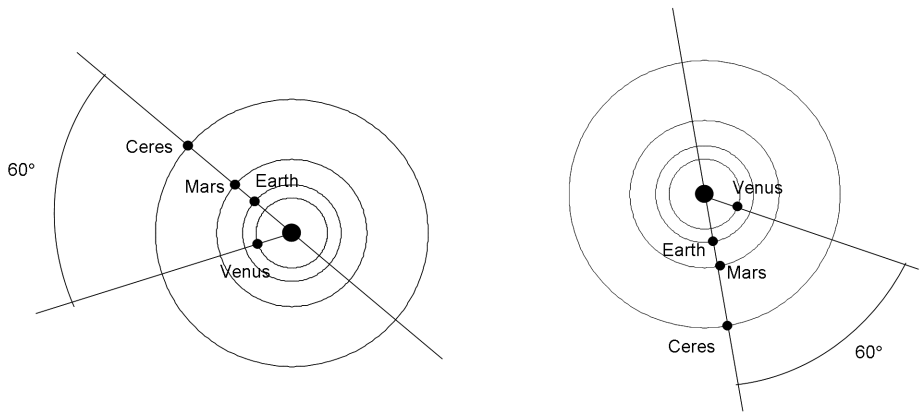

Fig. 2. Configuration of Venus, Earth, Mars and Ceres on 09.02.1995 and 02.07.2001

The process stops when $|\Delta P_1|/P_1 \leq 0.01$ for i and it is smaller than $|\Delta P_1|/P_1$ for $i+1$. In almost all resonances considered here the values of i or j are smaller or equal to 5.

Table 4. Synodic periods of the terrestrial planets, Ceres and Jupiter in days

Planet	Venus	Earth	Mars	Ceres	Jupiter
Mercury	144.5662183	115.8774777	100.8882096	92.8311021	89.7924087
Venus		583.9213707	333.9215172	259.4029420	236.9918953
Earth			779.9360970	466.7559931	398.8840447
Mars				1162.3977480	816.4345610
Ceres					2743.1291270

2 Three-planet resonances, including a terrestrial planet

In Table 4 the synodic periods of the terrestrial planets, the dwarf planet Ceres and Jupiter are given.

Among three-planet resonances presented in Table 5, two of the most accurate are Venus - Earth - Mars and Earth - Mars - Ceres. Both resonances have almost equal synodic period 6.4 *yr*. Therefore after a period equal to nine Venus - Ceres synodic periods the four bodies will reproduce their initial configuration (Fig.2) and according to (11) we obtain:

$$\frac{P_{VMa}}{P_{ECe}} \approx \frac{5}{7}. \tag{19}$$

The last equation can be written as follows:

$$5n_V - 7n_E - 5n_{Ma} + 7n_{Ce} \approx 0. \tag{20}$$

From the formulae that express the resonances Venus - Earth - Mars:

$$3n_V - 7n_E + 4n_{Ma} \approx 0 \tag{21}$$

and Earth - Mars - Ceres:

$$2n_E - 5n_{Ma} + 3n_{Ce} \approx 0 \tag{22}$$

are obtained relations between the mean longitudes of the bodies. At present these relations are:

$$\begin{aligned} 3\lambda_V - 7\lambda_E + 4\lambda_{Ma} &\approx 180^\circ, \\ 2\lambda_E - 5\lambda_{Ma} + 3\lambda_{Ce} &\approx 0^\circ. \end{aligned} \tag{23}$$

The first of these relations shows that the planets Venus, Earth and Mars can't lie in a line on the same side of the Sun (Fig.3). The second one shows that the planets Earth, Mars and dwarf planet Ceres can lie in a line on the same side of the Sun (Fig.2) In comparison with (23) the analogous relation of the Galilean satellites Io, Europa and Ganymede is:

$$\lambda_{Io} - 3\lambda_{Eu} + 2\lambda_{Ga} \approx 180^\circ. \tag{24}$$

Table 5. Three-planet resonances, including a terrestrial planet

Resonance	i:j	P_{ij} $[yr]$	φ [°]	ψ [′]	P_m $[yr]$	ΔP_1 $[dy]$	$\Delta P_1/P_1$ 10^{-3}
Venus, Earth, Mars	3:4	6.40	1	10	1059	0.15	1
Earth, Mars, Ceres	2:3	6.39	3	26	413	-1.10	-3
Venus, Earth, Ceres	5:4	6.39	1	6	1762	-0.07	0
Venus, Mars, Ceres	2:7	6.39	3	29	378	-0.82	-4
Mercury, Mars, Ceres	2:23	6.35	1	12	919	0.14	2
Mercury, Earth, Ceres	1:4	1.27	2	95	114	0.47	5
Mercury, Venus, Earth	1:4	1.59	3	106	102	0.52	6
Mercury, Venus, Mars	3:7	2.76	8	167	65	-0.54	-6
Mars, Ceres, Jupiter	3:7	22.35	9	23	471	4.60	7
Venus, Earth, Jupiter	3:2	3.24	11	195	55	2.10	9
Venus, Earth, Jupiter	41:28	44.77	2	2	4947	0.02	0
Mercury, Venus, Jupiter	3:5	1.97	11	341	32	-0.88	-10

Let's recall that the angles between three bodies in the partial solutions of the three-body problem are 0°, 60° or 180° and they doesn't depend on the masses of the bodies. Further partial solutions of the four-body problem using the perturbation theory approach could explain the relations between mean longitudes of Venus, Earth, Mars and Ceres.

From the equations (23) similar to (20) relations are obtained:

$$\lambda_V - 3\lambda_E + 3\lambda_{Ma} - \lambda_{Ce} \approx 60^\circ,$$
$$\lambda_V - \lambda_E - 2\lambda_{Ma} + 2\lambda_{Ce} \approx 60^\circ. \quad (25)$$

The first of these relations shows that after a period equal to three Venus - Ceres synodic periods (2.13 yr) Venus - Ceres and Mars - Earth in pairs will reproduce their configurations (Fig.4).

The second one shows that after the Mars - Ceres synodic period (3.2 yr) Venus - Earth and Mars - Ceres in pairs keep their configurations (Fig.5).

Some of the resonances in Table 5 are the mathematical consequence from others. For example: if the mean motions of Mars and Earth are excluded from the equations (21) and (22), then we obtain the equations that express the third and the forth resonance respectively.

If the mean motion of Earth is excluded from the equations (21) and from the equation that expresses the resonances Mercury - Venus - Earth:

$$n_M - 5n_V + 4n_E \approx 0 \quad (26)$$

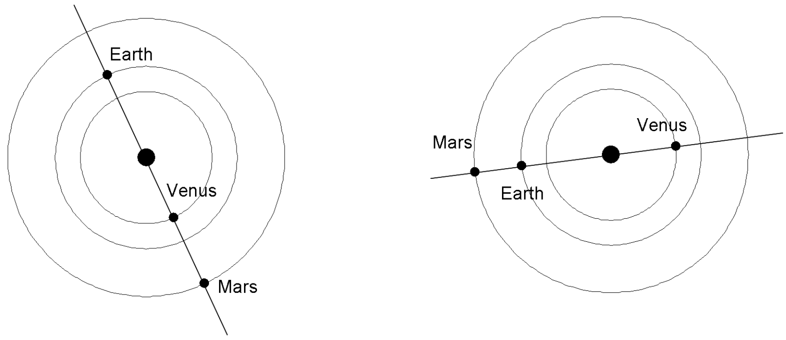

Fig. 3. Configuration of Venus, Earth and Mars on 15.01.1994 and 29.03.1997

An equation is derived that expresses a three-planet resonance between Mercury, Venus and Mars:

$$7n_M - 23n_V + 16n_{Ma} \approx 0. \tag{27}$$

This resonance has a synodic period 6.35 yr, which is approximately equal to the synodic periods of the resonances Venus - Earth - Mars and Earth - Mars - Ceres. Therefore the resonance Mercury - Venus - Mars (Table 5) is not in an agreement with other resonances and it must be considered as a coincidence.

The resonance Mercury - Mars - Ceres can be derived from the resonances Mercury - Venus - Earth, Venus - Earth - Mars and Earth - Mars - Ceres and it is only a mathematical consequence of upper resonances.

The three-planet resonance Venus - Earth - Jupiter (41:28) is taken from "Do Periodic Peaks in the Planetary Tidal Forces Acting Upon the Sun Influence the Sunspot Cycle?" [2011 b, Wilson]. It can be derived from the equations (21) and (22) and the formula that expresses the three-planet resonance Mars - Ceres - Jupiter if the mean motions of Mars and Ceres are excluded. It also can be derived from the equation for the four-planet resonance Venus - Earth - Mars - Jupiter:

$$5n_V - 5n_E - 7n_{Ma} + 7n_J \approx 0 \tag{28}$$

and the equation for the three-planet resonance Venus - Earth - Mars (21) if the mean motion of Mars is excluded. Therefore, as the three-planet resonance Venus - Earth - Jupiter (3:2) is less accurate and it is not in an agreement with other resonances it must be considered as a coincidence.

The last resonance in Table 5 is most inaccurate and it can't be derived as mathematical consequence from other. Because of that it must be considered as a coincidence too.

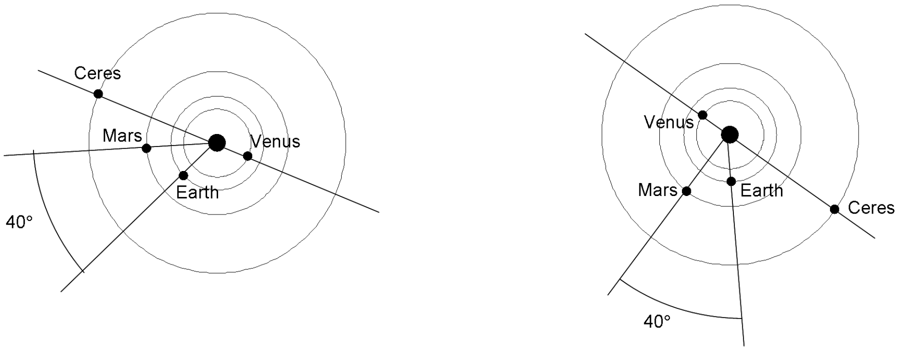

Fig. 4. Configuration of Venus, Earth, Mars and Ceres on 06.05.1995 and 23.06.1997

Therefore the resonances that have shortest times of mismatch Mercury - Venus - Mars, Venus - Earth - Jupiter (3:2) and Mercury - Venus - Jupiter (Table 5) must be excluded and must be considered as a coincidence. For the rest it can be said that each body has a three-planet resonance with its neighbors. The three-planet resonances, including Jupiter or Mercury, are inaccurate. The Mercury - Mars - Ceres synodic period is approximately equal to the synodic period of Earth - Mars - Ceres. The ratio of the synodic period of Mercury - Venus - Earth and that of Earth - Mars - Ceres is approximately equal to 1:4. And the ratio of the synodic period of Mars - Ceres - Jupiter and that of Earth - Mars - Ceres is approximately equal to 7:2 and the first one is approximately equal to a half of the Venus - Earth - Jupiter synodic period $P_{41,28}$. All that shows that there is a linear relationship between the mean motions of all terrestrial planets, the dwarf planet Ceres and the biggest planet in the Solar system - Jupiter. Our initial calculations on four-planet resonances show that they have period of repetition approximately equal to the last period 44.8 *yr*.

3 Three-planet resonances, including an asteroid from the main belt

All asteroids considered in this work are involved in three-planet resonances. The most accurate of them (Table 6) is Vesta - Juno - Ceres with synodic period 86.16 *yr* and time of mismatch 90631 *yr*.

Initially we took the values of the orbital periods of the asteroids from ASTORB Database [Asteroid Observing Services] and we calculated a deviation per year $\psi = 1'$ and time of mismatch 15912 *yr* for the resonance Astraea - Juno - Pallas. At present the relation between the mean longitudes

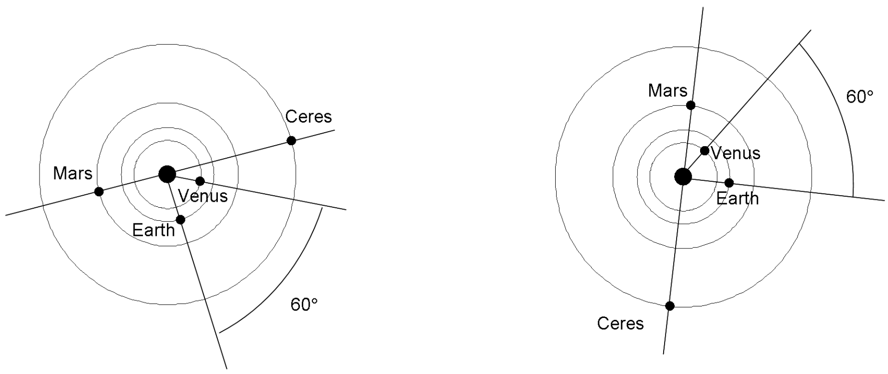

Fig. 5. Configuration of Venus, Earth, Mars and Ceres on 08.07.1993 and 12.09.1996

of Astraea, Juno and Pallas is:

$$\lambda_{As} - 2\lambda_{Ju} + \lambda_{Pa} \approx 60^\circ. \tag{29}$$

This relation is shown graphically in Fig. 6. For a time equal to 1/6 of the Astraea - Juno - Pallas synodic period Astraea will rotate by an angle of 60° relative to Juno and Pallas by -60°. Since the orbital period of Juno is approximately equal to 1/18 of Astraea - Juno - Pallas synodic period it will stay in the same place.

In comparison to (29) the analogous relation of the Galilean satellites Io, Europa and Ganymede is (24).

It is worth noting that the synodic periods of the three-planet resonances Mars - Flora - Hecuba and Flora - Hecuba - Hilda (Table 6) are approximately equal. The latter suggests that there is a correlation between the mean motions of the four bodies and it is very accurate. If the values for the orbital periods of the asteroids are taken from ASTORB Database [Asteroid Observing Services] the relation is:

$$n_{Ma} - n_{Fl} - 5n_{He} + 5n_{Hi} \approx 1''/yr. \tag{30}$$

It is interesting that this dependence includes Jupiter's resonant asteroids and Mars. It also shows that after a period 21.16 yr, two by two Mars - Hecuba and Flora - Hilda conserve their configurations and moreover - the deviation is **only 1 degree per 17 000 years and the time of mismatch is 3** Myr. Compared to it, the deviation of the Galilean satellites Io - Europa - Ganymede in Laplace resonance is **1 degree per 80 000 years and time of mismatch is 15** Myr. But the orbital periods of Galilean satellites Io and Ganymede are well known. They are approximately 400 times smaller than these of Mars and Hilda respectively. And Ganymede has revolved **20 500 times** since its discovery, but Hilda - **only 17 times**.

Table 6. Three-planet resonances, including an asteroid from the main belt

Resonance	i:j	P_{ij} $[yr]$	φ [°]	ψ [′]	P_m $[yr]$	ΔP_1 $[dy]$	$\Delta P_1/P_1$ 10^{-3}
Mars, Vesta, Jupiter	3:4	15.65	3	12	909	1.66	2
Vesta, Juno, Ceres	1:4	86.16	0	0	90631	0.13	0
Astraea, Juno, Pallas	1:1	78.38	10	8	1399	4.46	3
Astraea, Juno, Pallas *	1:1	78.08	1	1	15912	-0.39	0
Astraea, Juno, Ceres	8:9	687.82	8	1	16389	0.40	0
Vesta, Astraea, Juno	2:5	150.77	8	3	3329	2.53	2
Mars, Flora, Hecuba	3:5	22.20	1	4	2859	0.60	1
Flora, Hecuba, Hilda	1:3	22.23	1	1	7857	0.99	1
Hungaria, Alinda, Hecuba	3:4	35.43	6	10	1098	-2.85	-3

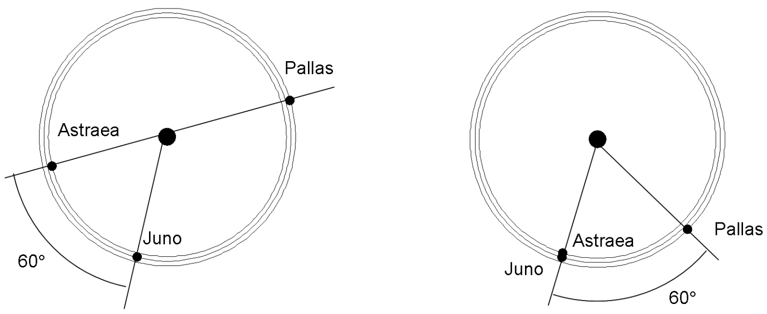

Fig. 6. Configuration of Astraea, Juno and Pallas on 26.07.2012 and 19.08.2025

4 Three-planet resonances, including a resonant asteroid from the main belt and a giant planet

Very accurate results for three-planet resonances, including a resonant asteroid from the main belt and a giant planet (Table 7) were expected.

3 times and 5 times more accurate resonances were obtained than the two-planet resonance for the asteroids Hilda and Thule respectively. If the orbital periods of the asteroids are taken from ASTORB Database [Asteroid Observing Services] then these three-planet resonances are 62 times and 40 times more accurate respectively. If the orbital period of Hilda is calculated

Table 7. Three-planet resonances, including a resonant asteroid from the main belt and a giant planet

Resonance	i:j	P_{ij} $[yr]$	φ [°]	ψ [']	P_m $[yr]$	ΔP_1 $[dy]$	$\Delta P_1/P_1$ 10^{-3}
Hilda, Jupiter, Saturn	6:5	119.07	2	1	13993	1.50	1
Thule, Jupiter, Saturn	7:4	138.75	5	2	5192	4.32	1
Hilda, Thule, Saturn	6:1	75.77	3	2	5254	-2.54	-1
Hilda, Thule, Jupiter	2:1	71.38	23	20	551	-30.80	-11
Hecuba, Thule, Jupiter	1:2	34.51	1	1	9267	2.02	1
Flora, Griqua, Jupiter	2:3	22.53	2	5	2000	-2.43	-2

from the two-planet resonance relation:

$$P_{Hi} = \frac{2}{3} P_J, \tag{31}$$

the result is 2888.39 *dy*. The value of the same period, derived from the formula for the three-planet resonance, including the orbital period of Saturn:

$$\frac{6}{P_{Hi}} - \frac{11}{P_J} + \frac{5}{P_S} = 0 \tag{32}$$

is 2892.71 *dy*. The orbital period of Hilda asteroid is 2892.78 *dy*. As for Thule asteroid corresponding values are: for the two-planet orbital resonance - 3249.44 *dy*, for the three-planet resonance - 3230.09 *dy*, the orbital period - 3230.56 *dy* and the formula for the three-planet resonance is:

$$\frac{7}{P_{Th}} - \frac{11}{P_J} + \frac{4}{P_S} = 0. \tag{33}$$

It is interesting that the middle coefficient in the formulae for the three-planet resonance of Hilda (26) and Thule (27) is the same -11. The first coefficients in both formulae are bigger than the last coefficients. If this formula is written with equal first and last coefficient:

$$\frac{1}{P} - \frac{2}{P_J} + \frac{1}{P_S} = 0, \tag{34}$$

orbital period of 7.43 *yr* can be derived. It is corresponding to a semimajor axis of 3.8 AU according to the third Kepler's law and it fits the end of Kirkwood gaps in the main asteroid belt.

If the orbital period of Saturn is excluded from the equations (33) and (34), an equation is derived that expresses a three-planet resonance between Hilda, Thule and Jupiter:

$$\frac{24}{P_{Hi}} - \frac{35}{P_{Th}} + \frac{11}{P_J} \approx 0. \tag{35}$$

This resonance has a synodic period 832.77 yr, that is 7 times bigger than the synodic periods of the resonance Hilda - Jupiter - Saturn, 6 times bigger than the synodic periods of the resonance Thule - Jupiter - Saturn and 11 times bigger than the synodic periods of the resonance Hilda - Thule - Saturn with very big accuracy. It has time of mismatch 4700 yr. Therefore the resonance Hilda - Thule - Jupiter (Table 7) is not in an agreement with other resonances and it must be excluded and considered as a coincidence.

The synodic period of the resonance Thule - Jupiter - Saturn is 4 times bigger than that of Hecuba - Thule - Jupiter. Therefore we can derive more relations between these four bodies. If the mean motion of Thule is expressed from (33) and put in the equation that expresses resonance Hecuba - Thule - Jupiter:

$$n_{He} - 3n_{Th} + 2n_J \approx 0, \tag{36}$$

An equation is derived that expresses a three-planet resonance Hecuba - Jupiter - Saturn:

$$7n_{He} - 19n_J + 12n_S \approx 0. \tag{37}$$

This resonance has a synodic period 138.39 yr, which is approximately equal of that of Thule - Jupiter - Saturn.

If the mean motion of Jupiter is expressed from (33) and put in (36) it is derived an equation that expresses a three-planet resonance Hecuba - Thule - Saturn:

$$11n_{He} - 19n_{Th} + 8n_S \approx 0 \tag{38}$$

with the same synodic period 138.39 yr. Therefore the four bodies are involved in a four-planet resonance.

5 Three-planet resonances, including a giant planet

Table 8 shows the synodic periods of Mercury, the giant planets and the dwarf planet Pluto.

Most interesting of the three-planet resonances, including a giant planet (Table 9), is the resonance Mercury - Jupiter - Neptune (39). It includes the largest, closest to the Sun and the most distant planet.

$$\frac{1}{P_M} - \frac{53}{P_J} + \frac{52}{P_N} \approx 0. \tag{39}$$

Table 8. Synodic periods of Mercury, the giant planets and the dwarf planet Pluto in days

Planet	Jupiter	Saturn	Uranus	Neptune	Pluto
Mercury	89.7924087	88.69443681	88.22214703	88.0980302	88.05474148
Jupiter		7253.452612	5044.814643	4668.693751	4550.150484
Uranus			16567.82345	13101.47317	12208.88356
Saturn				62620.01154	46404.54616
Neptune					179202.579

Table 9. Three-planet resonances, including a giant planet

Resonance	i:j	P_{ij} $[yr]$	φ [°]	ψ [']	P_m $[yr]$	ΔP_1 $[dy]$	$\Delta P_1/P_1$ 10^{-3}
Mercury, Jupiter, Neptune	1:52	12.78	0	0	59463	-0.01	0
Mercury, Jupiter, Saturn	1:81	19.91	1	3	3705	-0.23	-3
Mercury, Saturn, Neptune	1:148	35.94	1	1	9367	-0.17	-2
Jupiter, Saturn, Neptune	5:9	178.95	4	1	8048	8.96	2
Jupiter, Saturn, Uranus	4:9	179.56	15	5	2160	39.02	9
Jupiter, Uranus, Pluto	5:46	635.34	0	0	421857	-0.62	0
Saturn, Uranus, Pluto	5:14	635.10	0	0	273171	2.20	0
Jupiter, Saturn, Pluto	3:5	99.66	7	4	2699	25.54	6

The synodic period of this resonance is 12.8 yr. It is two times bigger than the synodic period of the three-planet resonances Venus - Earth - Mars and Earth - Mars - Ceres?! At present the relation between the mean longitudes[2] of Mercury, Jupiter and Neptune is:

$$\lambda_M - 53\lambda_J + 52\lambda_N \approx 240°. \qquad (40)$$

It is shown graphically in Fig. 7. When Jupiter and Neptune are in a syzygy Mercury is at 120° relative to Jupiter.

This resonance is unusual. It could be a consequence of an existence of a resonance between all terrestrial planets and the giant planets Jupiter and Neptune. This will be examined in a further work.

[2] The mean longitudes of the planets are taken from "Keplerian Elements for Approximate Positions of the Major Planets" (http://ssd.jpl.nasa.gov/txt/p_elem_t1.txt)

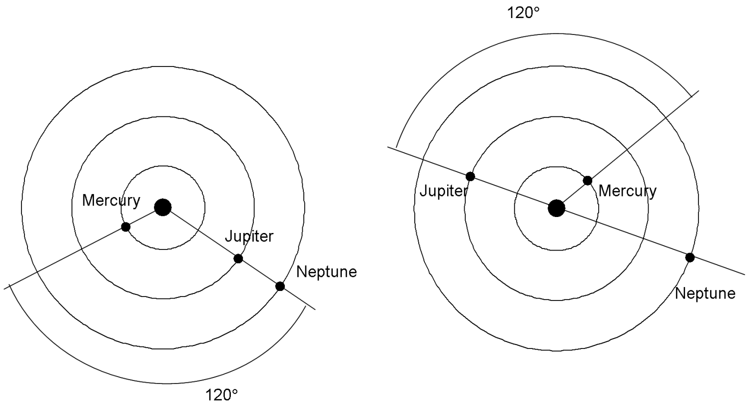

Fig. 7. Configuration of Mercury, Jupiter and Neptune on 09.08.2009 and 30.12.201 (The distances to the Sun are not scaled.)

The next resonance Mercury - Jupiter - Saturn is inaccurate. Its synodic period is approximately equal to 14/9 of the last period. More accurate resonance 9:727 with synodic period 178.72 *yr* and mismatch time 329358 *yr* was computed in a more detailed study.

The next resonance Mercury - Saturn - Neptune has approximately the same synodic period 179.45 *yr*. All this is in accordance with the next resonance Jupiter - Saturn - Neptune with a synodic period 178.95 *yr*. And even more, the next resonance Jupiter - Saturn - Uranus has almost the same synodic period 179.56 *yr*. If the mean motion of Jupiter is excluded from the equations that expressed these resonances a three-planet resonance Saturn - Uranus - Neptune (1:4) is derived with almost the same synodic period 179.35 *yr*. Therefore four giant planets are involved in a four-planet resonance with a period of repetition approximately equal to 179 *yr*.

The next two resonances have almost equal synodic periods of 635 *yr*. If the mean motion of Uranus is excluded from the equations that express these resonances an accurate three-planet resonance Jupiter - Saturn - Pluto (19:32) with the same synodic period and mismatch time 43633 *yr* is derived. Therefore the last resonance in Table 9 must be excluded and considered as a coincidence. By more detailed study some linear relations between the mean motions of Jupiter, Saturn, Uranus and Pluto can be derived. Most accurate of them is:

$$n_J - 4n_S + 5n_U - 2n_{Pl} \approx 1'/yr. \tag{41}$$

It can be written as:

$$P_{SUPl} = 2P_{JSU}, \tag{42}$$

where:

$$1/P_{JSU} = 1/P_{JS} - 1/P_{SU} \tag{43}$$

and

$$1/P_{SUPl} = 1/P_{SU} - 1/P_{UPl}. \qquad (44)$$

The last two periods can be named second synodic periods. Therefore the second synodic period of Saturn, Uranus and Pluto is twice the second synodic period of Jupiter, Saturn and Uranus. There is a similar statement for Venus, Earth, Mars and Ceres. The second synodic period of Venus, Earth and Mars is equal to the second synodic period of Earth, Mars and Ceres. Such type of resonances and the physical and geometrical properties of the second synodic periods will be a subject of further study.

Our initial calculations on four-planet resonances show a four-planet resonance Mercury - Jupiter - Saturn - Neptune:

$$\frac{3}{P_M} - \frac{247}{P_J} + \frac{247}{P_S} - \frac{3}{P_N} = 2'/yr. \qquad (45)$$

It has a synodic period 59.58 yr and time of mismatch 2.5 Myr. Most probably this resonance can also be derived from the equations that express resonances between all terrestrial planets and the giant planets. The period of repetition of giant planets is approximately three times bigger than this period. The ratio of the last period and the synodic period of the three-planet resonances Venus - Earth - Mars - Ceres and Earth - Mars - Ceres (Table 5) is approximately equal to 28:3 and the ratio of it and the synodic period of the three-planet resonance Venus - Earth - Jupiter $P_{41,28}$ (Table 5) is approximately equal to 4:3. Let us note that the last period is approximately equal to the period of repetition of all terrestrial planets, dwarf planet Ceres and Jupiter. Therefore all planets and Ceres may have a period of repetition approximately equal to 179 yr. This will be investigated in a further work.

Conclusion

In this work it was shown that there are many three-planet resonances in the Solar system. In general they are more in number and more accurate than the two-planet orbital resonances. The majority of the most accurate among them Venus - Earth - Mars (21), Earth - Mars - Ceres (22), Hilda - Jupiter - Saturn (32) and Mercury - Jupiter - Neptune (39) are expressed by adjacent integers, as well as the most accurate two-planet mean motion resonances Hilda - Jupiter (3:2), Thule - Jupiter (4:3) and Neptune - Pluto (3:2).

It was found that there are relations between the mean motions and the mean longitudes of four bodies. These four-planet resonances will be a subject of further work.

A detailed study of three-planet and four-planet resonances can be provided in order to explain more precisely the structure, origin and evolution of the Solar system. Further, particular solutions of the four-body problem using the perturbation theory approach could explain the nature of such resonances. Our initial calculations show that there is a relation between the mean motions of all terrestrial planets, dwarf planet Ceres and Jupiter and there is a relation between the mean motions of all giant planets and dwarf planet Pluto.

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