Galaxies, Cosmology and Dark Matter

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Chapter 4

Stellar and Galactic Dynamics

The dynamics of star clusters, galaxies or galaxy clusters is significantly more complicated than hydrodynamics. This has two basic reasons:

- Gases and plasmas (in laboratories, in stars) are dominated by electromagnetic forces which are mostly negligible on scales larger than a few times the typical separation of the particles.
  Galaxies are dominated by gravitation, a force that cannot be shielded. Therefore, stars/galaxies experience accelerations from all other members in the system.
- The mean free path of particles in most gases is generally small compared to the size of the system.
  In stellar and galaxy systems the mean free path is large compared to the size of the system ($\rightarrow$ few interactions, large relaxation times)

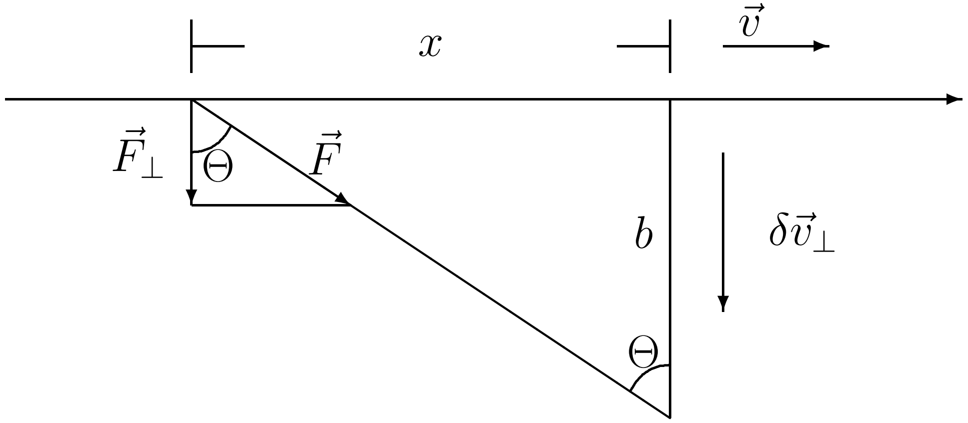
$\Rightarrow$ The physics of gases and plasmas is LOCAL
$\Rightarrow$ The physics of stellar systems is GLOBAL

For a comprehensive overview of stellar dynamics see:
Binney, Tremaine: Galactic Dynamics, Princeton Univ. Press

4.1 Relaxation of Stellar Systems

Classical relaxation is based on the redistribution of the orbital energies of stars via two-body encounters. After many encounters an equilibrium distribution is established comparable (but not equal!) to the Boltzmann distribution of statistical mechanics.

1. Deflection of a star when passing another star:



Consider passages at large distances first: $\delta v_\perp \ll v$

$$F_{\perp} = \frac{Gm^2}{b^2 + x^2} \cos(\Theta) = \frac{Gm^2 b}{(b^2 + x^2)^{3/2}} = \frac{Gm^2}{b^2} \left(1 + \left(\frac{x}{b}\right)^2\right)^{-3/2} \quad (4.1)$$

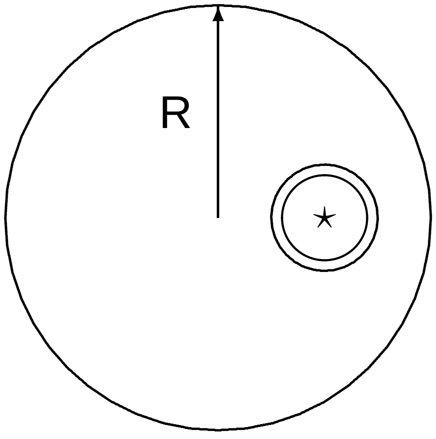
Setting the zero-point in t such that $x = v \cdot t$ gives:

$$\begin{aligned} \delta v_{\perp} &= \int_{-\infty}^{\infty} \frac{F_{\perp}}{m} dt = \int_{-\infty}^{\infty} \frac{Gm}{b^2} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-3/2} dt \\ &= \frac{Gm}{bv} \int_{-\infty}^{\infty} \left(1 + s^2\right)^{-3/2} ds \end{aligned} \quad (4.2)$$

$\Rightarrow$

$$\begin{aligned} \delta v_{\perp} &= \frac{2Gm}{bv} = \frac{Gm}{b^2} \cdot \frac{2b}{v} \\ &= (accelaration) \cdot (passage\ time) \end{aligned} \quad (4.3)$$

2. Number of interactions experienced by a star when passing through a stellar system once:



R = Radius of the stellar system
N = Number of stars in the system

Probability P_1 that the crossing star will pass one star of the system (e.g. galaxy) in a distance-interval $[b, b + db]$:

$$P_1 = \frac{2\pi b db}{\pi R^2} \tag{4.4}$$

If the galaxy contains N stars, the total number of interactions for a single crossing is:

$$\delta n_b = \frac{2N}{R^2} b db \tag{4.5}$$

With every interaction v changes by the amount of $\delta v_\perp$.
The sum of all interactions will lead to an average change of velocity of $<\delta v_\perp> \simeq 0$ (positive and negative deflections are equally probable).
However the mean square deflection is not equal to 0:

$$<\delta v_\perp^2> = \left(\frac{2Gm}{bv}\right)^2 \cdot \delta n_b = \left(\frac{2Gm}{bv}\right)^2 \cdot \frac{2Nb}{R^2} db \tag{4.6}$$

3. Integration over all impact parameters b:

$$<\delta v_\perp^2> = \int_{b_{min}}^{b_{max}} \left(\frac{2Gm}{bv}\right)^2 \frac{2Nb}{R^2}\, db = 8N \left(\frac{Gm}{vR}\right)^2 \ln\frac{b_{max}}{b_{min}} \tag{4.7}$$

Plausible values for b_{min}, b_{max}:

$b_{max} \simeq R$
$b_{min} \simeq \frac{Gm}{v^2}$ corresponding to $\delta v_\perp \simeq v$, i.e. the approximation of small deflection angles is no longer valid.

Using the virial theorem $|2T| = |V|$ gives:
($v =$ mean velocity of the stars)

$$2 \cdot \left(\frac{1}{2} N m v^2\right) = \frac{G(Nm)^2}{R} \tag{4.8}$$

$\Rightarrow$

$$\frac{Gm}{v^2} = \frac{R}{N} \tag{4.9}$$

and thus: $b_{min} \simeq \frac{R}{N}$.

As a matter of fact, interactions with $b < b_{min}$ are very rare:
The fractional area of a galaxy that corresponds to close passages is given by:

$$\frac{N\pi b_{min}^2}{\pi R^2} \simeq \frac{1}{N} \tag{4.10}$$

i.e. for typical stellar systems with $N > 10^5$ close interactions are negligible
$\Rightarrow$ Relaxation is dominated by large distance interactions

4. Using the virial theorem once more leads to

$$\frac{<\delta v_\perp^2>}{v^2} \simeq \frac{8NG^2m^2}{v^4R^2}\ln N = \frac{8}{N}\ln N \tag{4.11}$$

for a single passage through the stellar system.

For relaxation $(<\delta v_\perp^2> \simeq v^2)$ to occur, a star will have to cross the galaxy N_{relax} times:

$$\boxed{N_{relax} = \frac{N}{8\ln N}} \tag{4.12}$$

The relaxation-time τ_{relax} is:

$$\boxed{\tau_{relax} = N_{relax}\cdot\tau_{cross} = \frac{N}{8\ln N}\cdot\tau_{cross} \simeq \frac{N}{8\ln N}\cdot\frac{R}{v}} \tag{4.13}$$

($\tau_{cross} =$ crossing-time)

Examples:

	N	R	v	τ_{cross}	τ_{relax}	$\frac{age}{\tau_{relax}}$
open cluster	100	$2\,pc$	$0.5\,\frac{km}{s}$	$4\cdot 10^6\,yrs$	$10^7\,yrs$	≥ 1
globular cluster	10^5	$4\,pc$	$10\,\frac{km}{s}$	$4\cdot 10^5\,yrs$	$4\cdot 10^8\,yrs$	≥ 10
ellipt. galaxy	10^{12}	$10\,kpc$	$600\,\frac{km}{s}$	$2\cdot 10^7\,yrs$	$10^{17}\,yrs$	10^{-7}
dwarf galaxy	10^9	$1\,kpc$	$50\,\frac{km}{s}$	$2\cdot 10^7\,yrs$	$10^{14}\,yrs$	10^{-4}
galaxy cluster	1000	$1\,Mpc$	$1000\,\frac{km}{s}$	$10^9\,yrs$	$2\cdot 10^{10}\,yrs$	10^{-1}

$\Rightarrow$ Two-body-relaxation insignificant in galaxies and clusters of galaxies!

$\Rightarrow$ The velocity distribution in galaxies and galaxy clusters can be ANISOTROPIC!

i.e. at every location $\vec{x}$:

$$\overline{v_x^2} \neq \overline{v_y^2} \neq \overline{v_z^2} \tag{4.14}$$

can be true. This corresponds to an 'anisotropic temperature'.

Thus, the following applies to all galaxies:

- Stars do not experience significant encounters. Their orbital energy is largely preserved.
- The orbit of a star is determined by the smoothed gravitational potential of all other stars only.
- The density and velocity distribution of a stellar system can be approximated by a

phase-space distribution function $f\,(\vec{x}, \vec{v}, t)$

$f\,(\vec{x}, \vec{v}, t)\, d^3x\, d^3v$ = Fraction of the stars within the volume d^3x around the location $\vec{x}$ with velocities in the interval $[\vec{v}, \vec{v} + d\vec{v}]$

$\int f d^3v = n(\vec{x}, t)$ = Number density

- The time evolution of $f\,(\vec{x}, \vec{v}, t)$ is determined by Newtonian dynamics.
- Since stars can neither be created nor destroyed, a continuity equation for $f\,(\vec{x}, \vec{v}, t)$ exists:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \frac{\partial (f \dot{w}_i)}{\partial w_i} = 0, \quad (w_i = (x_1, x_2, x_3, v_1, v_2, v_3)) \tag{4.15}$$

From $\dot{x}_j = v_j$, $\dot{v}_j = -\frac{\partial \Phi}{\partial x_j}$ and

$$\sum \frac{\partial \dot{w}_i}{\partial w_i} = \sum \frac{\partial v_j}{\partial x_j} + \sum \frac{\partial}{\partial v_j}\left(-\frac{\partial \Phi}{\partial x_j}\right) = 0 \tag{4.16}$$

$\sum \frac{\partial v_j}{\partial x_j}$ = 0 since v_j and x_j are independent

$\sum \frac{\partial}{\partial v_j}\left(-\frac{\partial \Phi}{\partial x_j}\right)$ = 0 since Φ is independent of v_j for the case of gravitational interaction

one obtains:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f}{\partial w_i} = 0 \tag{4.17}$$

which is called the collisionless Boltzmann equation.

4.2 The Collisionless Boltzmann Equation

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \cdot \frac{\partial f}{\partial v_i} = 0} \tag{4.18}$$

Basic equation of stellar dynamics = continuity equation for the phase-space density

Important:

- So far, no assumption has been made as to whether or not the potential Φ is only due to the particles themselves or has further contributions from other sources.
  If the potential is only due to the particles described by f, then self-consistency is fulfilled:

$$\boxed{\Delta\Phi = 4\pi G\rho = 4\pi G m \int f(\vec{x}, \vec{v}, t)\, d^3v} \tag{4.19}$$

$m \int f(\vec{x}, \vec{v}, t)\, d^3v = m \cdot n = \rho$ m = typical mass of a star
n = number density

- The full determination of $f\,(\vec{x}, \vec{v}, t)$ is practically impossible.
  Therefore, the comparison of models and observations usually relies on the moments of the collisionless Boltzmann equation:

  e.g.:

  number density: $n(\vec{x}, t) = \int f\,(\vec{x}, \vec{v}, t)\, d^3v$ = 0th moment in v

  mean velocity: $\overline{v_i}(\vec{x}, t) = \frac{1}{n} \int v_i f\,(\vec{x}, \vec{v}, t)\, d^3v$ = 1st moment in v

4.3 The Jeans Equations

4.3.1 0th Moment of the Boltzmann Equation in v

$$\int \frac{\partial f}{\partial t} d^3v + \int v_i \frac{\partial f}{\partial x_i} d^3v - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3v = 0 \tag{4.20}$$

$\longrightarrow$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i} \int v_i f d^3v - \frac{\partial \Phi}{\partial x_i} \int \left[f\left(v_i\right)\right]_{-\infty}^{\infty} d^2v_{\neq i} = 0 \tag{4.21}$$

with $f\left(v_i = \pm\infty\right) = 0$ (4.21) gives:

$$\boxed{\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}\left(n\overline{v_i}\right) = 0} \tag{4.22}$$

1st Jeans equation, $\hat{=}$ continuity equation

4.3.2 1st Moment of the Boltzmann Equation in v

$$\int \frac{\partial f}{\partial t} v_j d^3v + \int v_i v_j \frac{\partial f}{\partial x_i} d^3v - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3v = 0 \tag{4.23}$$

With

$$\begin{aligned} \iiint v_j \frac{\partial f}{\partial v_i} d^3v \; &= \; \iint v_j \left(f\left(v_i = \infty\right) - f\left(v_i = -\infty\right)\right) d^2v_{\neq i} - \\ & \iiint \left(\frac{\partial v_j}{\partial v_i}\right) f d^3v = 0 - \delta_{ij} n \end{aligned} \tag{4.24}$$

Since: $f\left(v_i = \pm\infty\right) = 0$.

Thus:

$$\boxed{\frac{\partial}{\partial t}\left(n\overline{v_j}\right) + \frac{\partial}{\partial x_i}\left(n\overline{v_i v_j}\right) + n\frac{\partial \Phi}{\partial x_j} = 0} \tag{4.25}$$

2nd Jeans equation

With $\boxed{\overline{v_i v_j} = \frac{1}{n}\int v_i v_j f d^3v}$, $\frac{\partial v_j}{\partial t} = 0$, $\frac{\partial v_i}{\partial x_j} = 0$.

$(4.25) - \overline{v_j}\,(4.22) \;\; \Rightarrow$

$$n\frac{\partial \overline{v_j}}{\partial t} - \overline{v_j}\frac{\partial\left(n\overline{v_i}\right)}{\partial x_i} + \frac{\partial\left(n\overline{v_i v_j}\right)}{\partial x_i} + n\frac{\partial \Phi}{\partial x_j} = 0 \tag{4.26}$$

Velocity dispersion tensor σ_{ij}^2 :

$$\boxed{\sigma_{ij}^2 = \overline{\left(v_i - \overline{v_i}\right)\cdot\left(v_j - \overline{v_j}\right)} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j} \tag{4.27}$$

σ_{ij}^2 is the scattering of the velocities with respect to the mean velocities.

$$\frac{\partial\left(n\sigma_{ij}^2\right)}{\partial x_i} = \frac{\partial\left(n\overline{v_i v_j}\right)}{\partial x_i} - \overline{v_j}\frac{\partial\left(n\overline{v_i}\right)}{\partial x_i} - n\overline{v_i}\frac{\partial \overline{v_j}}{\partial x_i} \tag{4.28}$$

(4.26) and (4.28) $\Rightarrow$

$$\boxed{n\frac{\partial \overline{v_j}}{\partial t} + n\overline{v_i}\frac{\partial \overline{v_j}}{\partial x_i} = -n\frac{\partial \Phi}{\partial x_j} - \frac{\partial\left(n\sigma_{ij}^2\right)}{\partial x_i}} \qquad (4.29)$$

<u>3rd Jeans equation</u>

$n\frac{\partial \overline{v_j}}{\partial t} + n\overline{v_i}\frac{\partial \overline{v_j}}{\partial x_i}$: derivative of v

$-n\frac{\partial \Phi}{\partial x_j} - \frac{\partial\left(n\sigma_{ij}^2\right)}{\partial x_i}$: force terms

For comparison: hydrodynamical <u>Euler equation</u>:

$$\boxed{\rho\frac{\partial \vec{v}}{\partial t} + \rho\left(\vec{v}\vec{\nabla}\right)\vec{v} = -\rho\vec{\nabla}\Phi - \vec{\nabla}p} \qquad (4.30)$$

Note the difference between the third Jeans equation and the Euler equation:
The pressure term in the Jeans equation is a tensor, whereas the one in the Euler equation is a scalar and thus isotropic.

σ_{ij}^2 is a symmetric tensor, i.e. there exists a choice for the local coordinate system in which σ_{ij}^2 has diagonal form. In this system σ_{11}, σ_{22} and σ_{33} are the semi-axes of the dispersion ellipsoid. In the case of isotropic velocity dispersion $\sigma_{11} = \sigma_{22} = \sigma_{33}$ and the third Jeans equation is identical to the Euler equation.

In general, the Jeans equations cannot be solved without ambiguities, because for stellar systemes there exists no analogue to the equation of state $p = p(\rho)$ in case of gases.

$\Rightarrow$ In order to solve a problem of stellar dynamics using the Jeans equations, it is often necessary to make assumptions concerning σ_{ij}. Only more recently, improved observational techniques allow to constrain the σ_{ij} for galaxies.

Example: Anisotropic, spherically symmetric galaxy

In polar coordinates (r, θ, ϕ) and (v_r, v_θ, v_ϕ) spherical symmetry implies:

$$\overline{v_r} = \overline{v_\theta} = \overline{v_\phi} = 0 \tag{4.31}$$

$$\overline{v_\theta^2} = \overline{v_\phi^2} \tag{4.32}$$

In the stationary case one obtains:

$$\frac{dn\overline{v_r^2}}{dr} + \frac{n}{r}\left(2\overline{v_r^2} - \left(\overline{v_\theta^2} + \overline{v_\phi^2}\right)\right) = -n\frac{d\Phi}{dr} \tag{4.33}$$

Define the so-called **anisotropy parameter** β via:

$$\beta = 1 - \frac{\overline{v_\theta^2}}{\overline{v_r^2}} \tag{4.34}$$

then (4.33) becomes:

$$\boxed{\frac{dn\overline{v_r^2}}{dr} + 2n\beta\frac{\overline{v_r^2}}{r} = -n\frac{d\Phi}{dr}} \tag{4.35}$$

(This is equivalent to: $\frac{dp}{dr} +$ *anisotropy correction* $= -\rho g$ in hydrostatics)

The connection between the potential Φ, the circular velocity v_c at radius r and the mass $M(r)$ within a sphere of radius r, is given by:

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2} = \frac{v_c^2}{r} \tag{4.36}$$

i.e.

$$\boxed{v_c^2 = \frac{GM(r)}{r} = -\overline{v_r^2}\left(\frac{d\ln n}{d\ln r} + \frac{d\ln\overline{v_r^2}}{d\ln r} + 2\beta\right)} \tag{4.37}$$

comparison with hydrodynamics:

$-\overline{v_r^2}$	$\propto$	temperature
$\frac{d\ln n}{d\ln r} + \frac{d\ln \overline{v_r^2}}{d\ln r}$	:	as in hydrodynamics
2β	:	anisotropy correction

Example: Isotropic, axially symmetric galaxy

Here: $\rho = n$

$$\frac{\partial \rho \overline{v_r}}{\partial t} + \frac{\partial \rho \overline{v_r^2}}{\partial r} + \frac{\partial \rho \overline{v_r v_z}}{\partial z} + \rho \left(\frac{\overline{v_r}^2 - \overline{v_\phi}^2}{r} + \frac{\partial \Phi}{\partial r} \right) = 0 \quad (4.38)$$

$$\frac{\partial \rho \overline{v_\phi}}{\partial t} + \frac{\partial \rho \overline{v_r v_\phi}}{\partial r} + \frac{\partial \rho \overline{v_\phi v_z}}{\partial z} + \frac{2\rho}{r} \overline{v_\phi v_r} = 0 \quad (4.39)$$

$$\frac{\partial \rho \overline{v_z}}{\partial t} + \frac{\partial \rho \overline{v_r v_z}}{\partial r} + \frac{\partial \rho \overline{v_z^2}}{\partial z} + \frac{\rho \overline{v_r v_z}}{r} + \rho \frac{\partial \Phi}{\partial z} = 0 \quad (4.40)$$

Jeans equations in cylindrical coordinates

Consider a static, axially symmetric object with isotropic velocity dispersion:
(corresponding to a rotating, self-gravitating gas or liquid)

$$\overline{v_z^2} = \overline{v_r^2} = \overline{v_\phi^2} = \sigma^2 \quad (4.41)$$

All other components of $\sigma_{ij} = 0$, $(i \neq j)$

Furthermore the axial symmetry gives:

$$\overline{v_\phi} = 0 \text{ or } \overline{v_\phi} \neq 0 \tag{4.42}$$

$$\overline{v_r} = \overline{v_z} = 0 \tag{4.43}$$

This results in the following single non-trivial equation:
(In analogy to the barometric formula)

$$\frac{\partial \rho \sigma^2}{\partial r} - \rho \frac{\overline{v_\phi}^2}{r} + \rho \frac{\partial \Phi}{\partial r} = 0 \tag{4.44}$$

$\frac{\partial \rho \sigma^2}{\partial r}$: pressure gradient

$\rho \frac{\overline{v_\phi}^2}{r}$: centrifugal force

$\rho \frac{\partial \Phi}{\partial r}$: gravity

$$\frac{\partial \rho \sigma^2}{\partial z} - \rho \frac{\partial \Phi}{\partial z} = 0 \tag{4.45}$$

$\frac{\partial \rho \sigma^2}{\partial z}$: pressure gradient

$\rho \frac{\partial \Phi}{\partial z}$: gravity

Choose the following density distribution:

$$\rho = \rho_0 \left(r^2 + \frac{z^2}{q^2} \right)^{-\frac{K}{2}}, \quad K > 1 \tag{4.46}$$

Thus in the equatorial plane $(z = 0)$ the equations (4.44) and (4.45) reduce to:

$$-\frac{K}{r} \rho \sigma^2 + \rho \frac{\partial \sigma^2}{\partial r} = \rho \frac{\overline{v_\phi}^2}{r} - \rho \frac{\partial \Phi}{\partial r} \tag{4.47}$$

$\Rightarrow$

$$K\sigma^2 + \overline{v_\phi}^2 = v_c^2 + r\frac{\partial \sigma^2}{\partial r}, \quad \left(v_c^2 = r\frac{\partial \Phi}{\partial r}\right) \tag{4.48}$$

$\Rightarrow$

$$\boxed{v_c^2 \geq K\sigma^2 + \overline{v_\phi}^2} \tag{4.49}$$

4.4 The Virial Equations

For global considerations one generally considers the tensor-virial-theorem. It is obtained from the first moment of the 2nd Jeans equation (4.25) in the spatial coordinates.

$$m \cdot \int x_k \left[\text{2nd Jeans equation}\right] d^3x \quad \Rightarrow \quad \text{tensor-virial-theorem}$$
($m =$ mass of a single star, thus $m \cdot n = \rho$)

$\longrightarrow$

$$\int x_k \frac{\partial \left(\rho \overline{v_j}\right)}{\partial t} d^3x = -\int x_k \frac{\partial}{\partial x_i} \left(\rho \overline{v_i v_j}\right) d^3x - \int x_k \rho \frac{\partial \Phi}{\partial x_j} d^3x \tag{4.50}$$

- 3rd expression:

$$\boxed{W_{jk} = -\int x_j \rho \frac{\partial \Phi}{\partial x_k} d^3x} \tag{4.51}$$

Potential energy tensor

Replacing $\Phi = -G\int \frac{\rho(\vec{x'})}{\left|\vec{x}-\vec{x'}\right|} d^3x'$ gives:

$$W_{jk} = G\iint x_j \rho\left(\vec{x}\right)\rho\left(\vec{x'}\right)\frac{\left(x'_k - x_k\right)}{\left|\vec{x'}-\vec{x}\right|^3} d^3x d^3x' \tag{4.52}$$

The integration variables $\vec{x}$ and $\vec{x'}$ can be exchanged without changing the result of the integration:

$$W_{jk} = -G\iint x'_j \rho\left(\vec{x'}\right)\rho\left(\vec{x}\right)\frac{\left(x'_k - x_k\right)}{\left|\vec{x'}-\vec{x}\right|^3} d^3x' d^3x \tag{4.53}$$

Adding (4.52) and (4.53) and dividing the result by 2 gives:

$$\boxed{W_{jk} = -\frac{G}{2}\iint \rho\left(\vec{x}\right)\rho\left(\vec{x'}\right)\frac{\left(x'_j - x_j\right)\left(x'_k - x_k\right)}{\left|\vec{x'}-\vec{x}\right|^3} d^3x' d^3x} \tag{4.54}$$

Thus

$$\boxed{W_{jk} = W_{kj}} \tag{4.55}$$

and

$$\begin{aligned}
\boxed{\text{trace } W_{jk}} &= -\frac{G}{2}\iint \rho(\vec{x})\,\rho\left(\vec{x'}\right)\frac{\sum\left(x'_j - x_j\right)^2}{\left|\vec{x'} - \vec{x}\right|^3} d^3x' d^3x \\
&= -\frac{G}{2}\iint \frac{\rho(\vec{x})\,\rho\left(\vec{x'}\right)}{\left|\vec{x'} - \vec{x}\right|} d^3x' d^3x \\
&= \frac{1}{2}\int \rho\Phi d^3x \\
&\equiv \boxed{W}
\end{aligned} \tag{4.56}$$

$$\boxed{W = \text{Potential energy}}$$

- 1st expression:

$$\int x_k \frac{\partial \left(\rho \overline{v_j}\right)}{\partial t} d^3x \tag{4.57}$$

In order to transform this expression, it is necessary to determine the second derivative in time of the moment of inertia tensor.

$$I_{jk} = \int \rho x_j x_k d^3x \tag{4.58}$$

Moment of inertia tensor

$$\frac{d}{dt}I_{jk} = \int \frac{\partial \rho}{\partial t} x_j x_k d^3x \quad \text{using the continuity equation (4.22)} \tag{4.59}$$
$$= -\int \frac{\partial\left(\rho \overline{v_i}\right)}{\partial x_i} x_j x_k d^3x \quad \text{divergence theorem}$$
$$= \int \rho \overline{v_i}\left(\delta_{ij} x_k + \left(\delta_{ik} x_j\right) d^3x\right.$$
$$= \int \rho\left(\overline{v_j} x_k + \overline{v_k} x_j\right) d^3x$$

$$\frac{d^2}{dt^2} I_{jk} = \int x_k \frac{\partial}{\partial t}\left(\rho \overline{v_j}\right) + x_j \frac{\partial}{\partial t}\left(\rho \overline{v_k}\right) d^3x \tag{4.60}$$

Except for the symmetrization and a factor 2, this expression is identical to (4.57). (4.57) has to be symmetric, too, since the 3rd and the 2nd expression in (4.50) are symmetric.

Thus:

$$\boxed{\int x_k \frac{\partial\left(\rho \overline{v_j}\right)}{\partial t} d^3x = \frac{d^2}{dt^2}\left(\frac{1}{2} I_{jk}\right)} \tag{4.61}$$

2nd expression:

$$-\int x_k \frac{\partial}{\partial x_i} \left(\rho \overline{v_i v_j}\right) d^3x \tag{4.62}$$

Using the divergence theorem

$$\int_V g \vec{\nabla} \vec{F} d^3x = \int_{\partial V} g \vec{F} d^2\vec{s} - \int_V \left(\vec{F} \cdot \vec{\nabla}\right) g d^3x \tag{4.63}$$

expression (4.62) transforms into:

$$\begin{aligned} \boxed{-\int x_k \frac{\partial}{\partial x_i} \left(\rho \overline{v_i v_j}\right) d^3x} &= -\int_{\partial V} x_k \rho \overline{v_i v_j} d^2x_{\neq i} + \int_V \rho \overline{v_i v_j} \delta_{ik} d^3x \\ &= \int_V \rho \overline{v_k v_j} d^3x \\ &= \boxed{2K_{kj}} \end{aligned} \tag{4.64}$$

$\boxed{K_{kj} = \text{Kinetic energy tensor}}$

With $\overline{v_k v_j} = \bar{v}_k \bar{v}_j + \sigma_{kj}^2$ the kinetic energy tensor can be written as:

$$\begin{aligned} K_{jk} &= \int \frac{1}{2}\rho \bar{v}_j \bar{v}_k d^3x + \int \frac{1}{2}\rho \sigma_{jk}^2 d^3x \\ &= T_{jk} + \frac{1}{2}\Pi_{jk} \end{aligned} \quad (4.65)$$

T_{jk} : ordered motion
$\frac{1}{2}\Pi_{jk}$: random motion

$$\begin{aligned} \boxed{\text{trace } K_{jk}} &= \int \frac{1}{2}\rho \left|\vec{\bar{v}}\right|^2 d^3x + \int \frac{1}{2}\rho \left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2\right) d^3x \\ &\equiv \boxed{K} \end{aligned} \quad (4.66)$$

$\boxed{K = \text{Total kinetic energy}}$

Combining the three expressions results in the

Tensor Virial Theorem:

$$\frac{1}{2}\frac{d^2}{dt^2}I_{ik} = 2T_{ik} + \Pi_{ik} + W_{ik} \tag{4.67}$$

With

$$I_{ik} = \int \rho x_i x_k d^3x \qquad \text{Moment of inertia tensor} \tag{4.68}$$

$$T_{ik} = \int \frac{1}{2}\rho \bar{v}_i \bar{v}_k d^3x \qquad \text{Motion tensor} \tag{4.69}$$

$$\Pi_{ik} = \int \rho \sigma_{ik}^2 d^3x \qquad \text{Dispersion Tensor} \tag{4.70}$$

$$W_{ik} = -\frac{G}{2}\iint \rho\left(\vec{x}\right)\rho\left(\vec{x}'\right)\frac{\left(x_i' - x_i\right)\left(x_k' - x_k\right)}{\left|\vec{x}' - \vec{x}\right|^3}d^3x' d^3x \qquad \text{Potential energy tensor} \tag{4.71}$$

For the static case $\frac{d^2}{dt^2} I_{ik} = 0$, the trace reduces to:

$$\boxed{2T + \Pi = 2K = -W} \tag{4.72}$$

This is the well-known scalar virial theorem.

Comment: The tensor virial theorem describes a relation between global mean parameters of a stellar system. It is only valid for the system as a whole and not for any subsystems.